For Online Publication
Appendix to "Network-motivated Forbearance Lending"

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Abstract

This online appendix includes supplemental materials for "Network-motivated Forbearance Lending" by Ogura, Okui, and Saito. Online Appendix 1 contains the proofs of Propositions 4, 5, 6, and 7. Online Appendix 2 presents the details of product demand from outside. Online Appendix 3 contains the derivation for the typical network example in Section 3.2. Online Appendix 4 reports a numerical example.

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Online Appendix 1 Proofs of Propositions 4, 5, 6, and 7

Proof of Proposition 4. We have

$$\frac{1}{\theta}\mathbf{e}'_{b,-z}\mathbf{P}_\theta\mathbf{A}^{-1}\mathbf{\Phi}(\mathbf{e}-\mathbf{e}_{-z}) = \mathbf{e}'_{b,-z}\mathbf{P}_\theta\mathbf{A}^{-1}\hat{\mathbf{P}}_\theta\mathbf{P}_\theta\frac{1}{\theta}\mathbf{\Phi}(\mathbf{e}-\mathbf{e}_{-z}).$$

We have

$$\frac{1}{\theta}\mathbf{P}_\theta\mathbf{\Phi}(\mathbf{e}-\mathbf{e}_{-z}) = \frac{1}{\theta}p_z^{-(\theta-1)}\phi_z\mathbf{u}_z.$$

We have

$$\begin{aligned}\mathbf{e}'_{b,-z}\mathbf{P}_\theta\mathbf{A}^{-1}\hat{\mathbf{P}}_\theta =&\mathbf{e_b}'\mathbf{P}_\theta\mathbf{A}^{-1}\hat{\mathbf{P}}_\theta - (\mathbf{e_b}-\mathbf{e}_{b,-z})'\mathbf{P}_\theta\mathbf{A}^{-1}\hat{\mathbf{P}}_\theta \\ =&\mathbf{v_b}' - p_z^{-(\theta-1)}\mathbf{u}'_z\mathbf{A}^{-1}\hat{\mathbf{P}}_\theta.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{1}{\theta}\mathbf{e}'_{b,-z}\mathbf{P}_\theta\mathbf{A}^{-1}\mathbf{\Phi}(\mathbf{e}-\mathbf{e}_{-z}) \\ =&v_z\frac{1}{\theta}p_z^{-(\theta-1)}\phi_z - p_z^{-(\theta-1)}\mathbf{u}'_z\mathbf{A}^{-1}\hat{\mathbf{P}}_\theta\frac{1}{\theta}p_z^{-(\theta-1)}\phi_z\mathbf{u}_z \\ =&\left(v_z - A_{zz}^{(-1)}\right)\frac{1}{\theta}p_z^{-(\theta-1)}\phi_z,\end{aligned}$$

where $A_{zz}^{(-1)}$ is the (z,z)-th element of $\mathbf{A}^{-1}$. Note that $A_{zz}^{(-1)} = 1$ if either the z-th row of the z-th column of $\mathbf{W}$ is zero. Another situation in which $A_{zz} = 1$ occurs is that $\mathbf{W}$ is an upper triangular matrix, which is shown in the following lemma. □

Lemma 6. *Let* $\mathbf{A}$ *be a square upper triangular matrix. Suppose also that* $\mathbf{A}$ *is invertible and all the diagonal elements are 1. Then* $\mathbf{A}^{-1}$ *is upper triangular and its diagonal elements are all equal to 1.*

Proof. We prove by mathematical induction. Consider the following 2×2 matrix

$$\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix},$$

where a is a some real number. The inverse of this matrix is

$$\begin{pmatrix}1 & -a\\ 0 & 1\end{pmatrix}.$$

This is an invertible matrix. Suppose that the inverse of an $n\times n$ triangular matrix whose diagonal elements are all one is also an $n\times n$ triangular matrix whose diagonal elements are all one. Let $\mathbf{A}$

be such a matrix. Consider the following matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{a} \\ \mathbf{0}_n & 1 \end{pmatrix},$$

where $\mathbf{a}$ is an $n \times 1$ vector and $\mathbf{0}_n$ is a $1 \times n$ vector of zeros. The inverse of this matrix is

$$\begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{a} \\ \mathbf{0}_n & 1 \end{pmatrix}.$$

This matrix is a triangular matrix whose diagonal elements are all 1. □

Next, we provide the proofs of Propositions 6 and 7. For them, we use the following lemma.

Lemma 7. *Suppose that* $\mathbf{A}$ *is an* $n \times n$ *invertible matrix, and* $\mathbf{a}$ *and* $\mathbf{b}$ *are* $n \times 1$ *vectors. Suppose that* $\mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b} \neq 1$*. Then,*

$$\left(\mathbf{A} - \mathbf{a}\mathbf{b}'\right)^{-1} = \mathbf{A}^{-1} + \frac{1}{1 - \mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}\mathbf{A}^{-1}\mathbf{a}\mathbf{b}'\mathbf{A}^{-1}.$$

Proof.

$$\begin{aligned}&\left(\mathbf{A} - \mathbf{a}\mathbf{b}'\right)\left(\mathbf{A}^{-1} + \frac{1}{1 - \mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}\mathbf{A}^{-1}\mathbf{a}\mathbf{b}'\mathbf{A}^{-1}\right)\\ =&\mathbf{I} - \mathbf{a}\mathbf{b}'\mathbf{A}^{-1} + \frac{1}{1 - \mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}\mathbf{a}\mathbf{b}'\mathbf{A}^{-1} - \frac{\mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}{1 - \mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}\mathbf{a}\mathbf{b}'\mathbf{A}^{-1} = \mathbf{I},\end{aligned}$$

and

$$\begin{aligned}&\left(\mathbf{A}^{-1} + \frac{1}{1 - \mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}\mathbf{A}^{-1}\mathbf{a}\mathbf{b}'\mathbf{A}^{-1}\right)\left(\mathbf{A} - \mathbf{a}\mathbf{b}'\right)\\ =&\mathbf{I} - \mathbf{A}^{-1}\mathbf{a}\mathbf{b}' + \frac{1}{1 - \mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}\mathbf{A}^{-1}\mathbf{a}\mathbf{b}' - \frac{\mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}{1 - \mathbf{a}'(\mathbf{A}')^{-1}\mathbf{b}}\mathbf{A}^{-1}\mathbf{a}\mathbf{b}' = \mathbf{I}.\end{aligned}$$

□

Proof of Proposition 5 (First part) Suppose firm z does not use any inputs from other firms in the network, i.e., all the elements in the z-th row of $\mathbf{W}$ are zero. From the matrix expression of the demand influence vector (29):

$$\mathbf{v}_{\mathbf{b}}' = \mathbf{e}_{\mathbf{b}}'\mathbf{P}_\theta\mathbf{A}^{-1}\hat{\mathbf{P}}_\theta,$$

where

$$\mathbf{A}^{-1} = \left[\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}'\right]^{-1} = \sum_{k=0}^{\infty}\left(\frac{\theta-1}{\theta}\right)^{k\theta}(\mathbf{E}\mathbf{W}')^k.$$

From the definition of $\mathbf{A}^{-1}$, if all z-th row elements of $\mathbf{W}$, i.e., z-th column of $\mathbf{W}'$ are zero, the z-th column of $\mathbf{A}^{-1}$ is a unit vector whose z-th element is one and other elements are zero. Therefore, v_z is one by the above definition.

The demand propagation (26) is zero in this case because it is:

$$\frac{\phi_z}{\theta}\mathbf{p}_{\theta,-\mathbf{z}}{}'\mathbf{A}^{-1}\mathbf{u}_z = \frac{\phi_z}{\theta}\mathbf{p}_{\theta,-\mathbf{z}}{}'\mathbf{u}_z = 0$$

where $\mathbf{u_z}$ is a unit vector whose z-th element is one and the other elements are zero, and z-th element of $\mathbf{p}_{\theta,-\mathbf{z}}{}'$ is zero, and z-th column of $\mathbf{A}^{-1}$ is a unit vector whose z-th element is one and other elements are zero.

The additional link effect (27) is also zero because:

$$\mathbf{A}^{-1} = \sum_{k=0}^{\infty}\left(\frac{\theta-1}{\theta}\right)^{k\theta}(\mathbf{E}\mathbf{W}')^k = \sum_{k=0}^{\infty}\left(\frac{\theta-1}{\theta}\right)^{k\theta}(\mathbf{E}_{-\mathbf{z}}\mathbf{W}')^k = \mathbf{A}_{-\mathbf{z}}{}^{-1}.$$

The middle equality is a result of the assumption that z-th row elements of $\mathbf{W}$, i.e., z-th column elements of $\mathbf{W}'$, are zero.

(Second part) We observe, by Lemma 7,

$$\begin{aligned}
&\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{W}' - a\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{w}_{z\cdot}\mathbf{u}_z'\right)^{-1}\\
=&\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{W}'\right)^{-1}\\
&+\frac{1}{1-a\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{u}_z'\mathbf{E}\mathbf{w}_{z\cdot}}\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{W}'\right)^{-1}a\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{w}_{z\cdot}\mathbf{u}_z'\\
&\times\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{W}'\right)^{-1}
\end{aligned}$$

Assume that $e_z = 1$, otherwise this proposition is irrelevant. We have $\mathbf{E}\mathbf{u}_z = \mathbf{u}_z$. We also observe that $\mathbf{w}_{z,}'\mathbf{u}_z = 0$. Thus, we have

$$\begin{aligned}
&\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{W}' - a\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{u}_z\mathbf{w}_{z\cdot}'\right)^{-1}\\
=&\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{W}'\right)^{-1}\\
&+a\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{W}'\right)^{-1}\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{w}_{z\cdot}\mathbf{u}_z'\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{E}\mathbf{W}'\right)^{-1}.
\end{aligned}$$

When the z-th column of $\mathbf{W}$ is zero, then $\mathbf{u}_z' \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}'\right)^{-1} = \mathbf{u}_z'$. We then have

$$
\begin{aligned}
v_z' =& \mathbf{e}'\mathbf{P}_\theta \mathbf{A}^{-1} \hat{\mathbf{P}}_\theta \mathbf{u}_z \\
=& \mathbf{e}'\mathbf{P}_\theta \left(\left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}'\right)^{-1} + a \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}'\right)^{-1} \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{w}_{z\cdot}\mathbf{u}_z' \right) \mathbf{u}_z p_z^{\theta-1} \\
=& \mathbf{p}_\theta' \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}'\right)^{-1} \left(\mathbf{u}_z p_z^{\theta-1} + a \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{w}_{z\cdot} p_z^{\theta-1} \right) \\
=& 1 + a \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{p}_\theta' \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}'\right)^{-1} \left(\mathbf{w}_{z\cdot} p_z^{\theta-1}\right)
\end{aligned}
$$

By a similar calculation, we have

$$
\begin{aligned}
& \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}\mathbf{E} - a \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{u}_z \mathbf{w}_{z\cdot}' \mathbf{E}\right)^{-1} \\
=& \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{W}\mathbf{E}\right)^{-1} \\
& + \frac{1}{1 - a \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{w}_{,z}' \mathbf{E}\mathbf{u}_z} \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{W}\mathbf{E}\right)^{-1} a \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{w}_{z\cdot}\mathbf{u}_z' \mathbf{E} \\
& \times \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{W}\mathbf{E}\right)^{-1} \\
=& \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{W}\mathbf{E}\right)^{-1} \left(\mathbf{I} + a \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{w}_{z\cdot}\mathbf{u}_z'\right).
\end{aligned}
$$

By $\mathbf{p}_\theta = \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{w}_0 \mathbf{E}\mathbf{B}^{-1}$, we have

$$
\begin{aligned}
v_z =& 1 + a \left(\frac{\theta-1}{\theta}\right)^\theta \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{w}_0 \mathbf{E} \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{W}\mathbf{E}\right)^{-1} \\
& \times \left(\mathbf{I} + a \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{w}_{z\cdot}\mathbf{u}_z'\right) \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}'\right)^{-1} \left(\mathbf{w}_{z\cdot} p_z^{\theta-1}\right) \\
=& 1 + a \left(\frac{\theta-1}{\theta}\right)^\theta \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{w}_0 \mathbf{E} \left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{W}\mathbf{E}\right)^{-1} \\
& \times \left(\left(\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E}\mathbf{W}'\right)^{-1} + a \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}\mathbf{w}_{z\cdot}\mathbf{u}_z' \right) \left(\mathbf{w}_{z\cdot} p_z^{\theta-1}\right).
\end{aligned}
$$

We note that $p_z^{\theta-1}$ can be written as $a_p/(b_p + a)$ for some $a_p > 0$ and $b_p > 0$. Thus, the influence vector v_z has a form that $1 + (a + a_v a^2)/(b_v + c_v a)$ for some $a_v, b_v, c_v > 0$. Such a function is increasing in a. $\square$

Proof of Proposition 6 (First part) Suppose all elements in the z-th column of $\mathbf{W}$, i.e., the z-th row of $\mathbf{W}'$, are zero. The price externality component (28) is:

$$\frac{1}{\theta}\mathbf{e}_{\mathbf{b},-\mathbf{z}}{}'\left(\mathbf{P}_{\theta}-\mathbf{P}_{\theta,-\mathbf{z}}\right)\mathbf{A}_{-\mathbf{z}}{}^{-1}\mathbf{\Phi}\mathbf{e}_{-\mathbf{z}}=\frac{1}{\theta}\left(\mathbf{p}'_{\theta}-\mathbf{p}'_{\theta,-\mathbf{z}}\right)\mathbf{E}_{\mathbf{b},-\mathbf{z}}\mathbf{A}_{-\mathbf{z}}{}^{-1}\mathbf{\Phi}\mathbf{e}_{-\mathbf{z}}.$$

We can rewrite a part of this expression by using the matrix expression of $p_i^{1-\theta}$ in (10):

$$\begin{aligned}\left(\mathbf{p}'_{\theta}-\mathbf{p}'_{\theta,-\mathbf{z}}\right)\mathbf{E}_{\mathbf{b},-\mathbf{z}}&=\left(\mathbf{w}'_{\mathbf{0}}\mathbf{E}\mathbf{B}^{-1}-\mathbf{w}'_{\mathbf{0}}\mathbf{E}_{-\mathbf{z}}\mathbf{B}_{-\mathbf{z}}{}^{-1}\right)\mathbf{E}_{\mathbf{b},-\mathbf{z}}\\&=\mathbf{w}'_{\mathbf{0}}\left((\mathbf{E}-\mathbf{E}_{-\mathbf{z}})\mathbf{B}^{-1}-\mathbf{E}_{-\mathbf{z}}\mathbf{B}_{-\mathbf{z}}^{-\mathbf{1}}(\mathbf{B}_{-\mathbf{z}}-\mathbf{B})\mathbf{B}^{-\mathbf{1}}\right)\mathbf{E}_{\mathbf{b},-\mathbf{z}},\end{aligned}\tag{A.1}$$

where

$$\mathbf{B}^{-1}\equiv\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{E}\mathbf{W}'\mathbf{E}\right)^{-1}=\sum_{k=0}^{\infty}\left(\frac{\theta-1}{\theta}\right)^{(\theta-1)k}(\mathbf{E}\mathbf{W}'\mathbf{E})^{k}.$$

Note that all elements of z-th row of $\mathbf{B}^{-1}$ are zero if all elements in the z-th row of $\mathbf{W}'$ are zero. From the definition of $\mathbf{B}$:

$$\begin{aligned}\mathbf{B}_{-\mathbf{z}}-\mathbf{B}&=\left(\frac{\theta-1}{\theta}\right)^{\theta-1}(\mathbf{E}\mathbf{W}'\mathbf{E}-\mathbf{E}_{-\mathbf{z}}\mathbf{W}'\mathbf{E}_{-\mathbf{z}})\\&=\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\{(\mathbf{E}-\mathbf{E}_{-\mathbf{z}})\mathbf{W}'\mathbf{E}+\mathbf{E}_{-\mathbf{z}}\mathbf{W}'(\mathbf{E}-\mathbf{E}_{-\mathbf{z}})\}.\end{aligned}$$

Note that $\mathbf{E}-\mathbf{E}_{-\mathbf{z}}$ is a matrix whose elements are zero except that the z-th diagonal element is one. Plugging this into (A.1) and calculating it under the assumption that all elements in the z-th column in $\mathbf{W}$ are zero, we obtain the result that all elements of (A.1) are zero, and thus, the price externality (28) is zero.

Under the assumption that all elements in the z-th column in $\mathbf{W}$ are zero, the price influence coefficient ξ_z (30) takes the lowest possible value $\xi_z=1$, i.e., firm z does not have any impact on the product prices of other firms.

(Second part) We examine

$$\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{E}\mathbf{W}\mathbf{E}-a\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{E}\mathbf{w}_{,z}\mathbf{u}'_{z}\mathbf{E}\right)^{-1}.$$

By Lemma 7, we have

$$
\begin{aligned}
&\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{EWE}-a\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{Ew}_{,z}\mathbf{u}_z'\mathbf{E}\right)^{-1}\\
=&\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}\\
&+\frac{1}{1+a\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{w}_{,z}'\mathbf{Eu}_z'}\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}a\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{Ew}_{,z}\mathbf{u}_z'\mathbf{E}\\
&\times\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}
\end{aligned}
$$

Assume that $e_z=1$. We then have $\mathbf{u}_z'\mathbf{E}=\mathbf{u}_z'$. We also observe that $\mathbf{w}_{,z}'\mathbf{u}_z=0$. Thus, we have

$$
\begin{aligned}
&\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{EWE}-a\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{Ew}_{,z}\mathbf{u}_z'\mathbf{E}\right)^{-1}\\
=&\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}\\
&+\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}a\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{Ew}_{,z}\mathbf{u}_z'\\
&\times\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}.
\end{aligned}
$$

The supply influence coefficient under $\mathbf{W}-a\mathbf{w}_{,z}\mathbf{u}_z'$ is

$$
\begin{aligned}
&\mathbf{e}'\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}\\
&+a\mathbf{e}'\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{Ew}_{,z}\mathbf{u}_z'\\
&\times\left(\mathbf{I}-\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right)^{-1}.
\end{aligned}
$$

This indicates that the supply influence coefficient is increasing in a. □

Proof of Proposition 7 First, we calculate the value of each component of the externality (26)-(28) when θ is close to one, i.e., the least substitutable case Because $\lim_{\theta\to1}\{(\theta-1)/\theta\}^{\theta}=0$, $\mathbf{A}=\mathbf{A}_{-\mathbf{z}}=\mathbf{I}$, an identity matrix. Plugging this into the components of the demand propagation effect (26) and the additional-network effect (27) shows that these are zero. On the other hand, the price externality (28) remains positive because $\lim_{\theta\to1}\{(\theta-1)/\theta\}^{\theta-1}=1$.

Second, if $\theta \to \infty$, i.e., if products are perfectly substitutable, $\lim_{\theta\to\infty}\{(\theta-1)/\theta\}^{\theta} = 1/e$, and $\lim_{\theta\to 1}\{(\theta-1)/\theta\}^{\theta-1} = 1/e$, where e is the base of natural logarithm, i.e., each of them converges to a finite value, whereas $1/\theta$ at the left end of each term in the externality (26)-(28) goes to zero. Thus, if the elasticity of substitution θ goes to infinity, and the profit of each firm goes to zero, the externality disappears. □

Online Appendix 2 Details of the derivations in Section 2.2.2 "Demand from outside the network"

There are two types of demand for good i from outside a network. Firms outside the network demand good i as an input. Consumers also demand it as a final consumption good.

Input demand from firms outside the network. It is reasonable to assume that firms outside the network also determine their input demand by cost minimization in the same manner as the previous section. From the cost minimization, the intermediate demand for the i-th good from firms outside the network is expressed by:

$$\int_{j\notin N} e_j x_{ji} dj = \delta_i p_i^{-\theta}, \text{ where } \delta_i \equiv \int_{j\notin N} {p^j}^{\theta} w_{ji} e_j e_i x_j dj, \tag{A.2}$$

and N indicates the set of firms in the network. δ_i is treated as a non-negative constant by firm i in the network because a firm in the network cannot affect the average cost $p^j (j \notin N)$ and the output of firms $x_j (j \notin N)$ outside the network under the vast (continuum) variety of input goods in the entire economy.

Consumers. Another group of agents outside the network is consumers. We assume the utility-maximizing representative consumers, with a constant elasticity of substitution (CES) utility function under the budget constraint and the availability constraint.

Consumers choose their consumption of the continuum mass of m types of differentiated products by solving the following utility maximization problem,

$$\max_{c_i (i\in[0,m])} U = \left(\int_0^m \beta_i^{\frac{1}{\theta}} c_i^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}}, \quad \theta > 1, \text{s.t. } \int_0^m c_i p_i di \leq R,\ (1-e_i)c_i = 0,$$

where c_i is the consumption of good i, β_i is the importance of good i as a consumption good, θ is the elasticity of substitution, and R is the income of the consumer. The first constraint is the consumer's budget constraint. The second constraint is the availability constraint. This constraint implies that consumption has to be zero if the good is not supplied.

The utility maximization under these constraints gives the following consumers' demand for product $i \in [0, m]$.

$$c_i = \frac{e_i R \beta_i}{p_c}\left(\frac{p_c}{p_i}\right)^{\theta}, \text{ where } p_c \equiv \left(\int_0^m e_i \beta_i p_i^{1-\theta} di\right)^{\frac{1}{1-\theta}}.$$

Firm i in the network considers the income R, and the consumer price index p_c as exogenously given constants. Thus, in this case, $\gamma_i \equiv e_i R \beta_i p_c^{\theta-1}$ is treated as a constant by firm i.

Thus, the consumers' demand function for product i is similar to the outside intermediate demand (A.2)

$$c_i = \gamma_i p_i^{-\theta}, \tag{A.3}$$

where γ_i is a non-negative constant. γ_i is perceived as a constant by firm i because a firm in the network cannot affect the consumers' income and the consumer price index under the vast (continuum) variety of products in the entire economy.

Total demand for product i from outside the network. The sum of the input demand from firms outside the network (A.2) and the consumers' demand (A.3) is the demand for the i-th good from outside the network. We denote this outside demand by:

$$h_i = \phi_i p_i^{-\theta}, \text{ where } \phi_i \equiv \gamma_i + \delta_i.$$

ϕ_i is a non-negative constant.

Online Appendix 3 Details of the derivations in Section 3.2

This section presents the detailed derivations in Section 3.2. We first recall the general formulas, and present useful tricks for matrix calculation. We then show the derivations for each of two examples.

A3.1 Preliminary

The first part of this section, we recall the general formulas for profits and other quantities. In the second part, some useful lemmas for matrix algebra are presented.

Recall that

$$\mathbf{A} = \mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta} \mathbf{EW}'$$

and

$$\mathbf{B} = \mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{EW}'\mathbf{E}.$$

The price vector is

$$\mathbf{p}_\theta{}' = \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{w_0}\mathbf{E}\mathbf{B}^{-1}$$

The decomposition of the monopolistic bank profit is

$$\begin{aligned}\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}}) =& \frac{1}{\theta}\mathbf{e}'\mathbf{P}_\theta\mathbf{A}^{-1}\mathbf{\Phi}\mathbf{e} - \frac{1}{\theta}\mathbf{e_{-z}}'\mathbf{P}_{\theta,-\mathbf{z}}\mathbf{A_{-z}}^{-1}\mathbf{e_{-z}} \\ =& \pi_z(\mathbf{e}) \\ &+ \frac{1}{\theta}\mathbf{e_{-z}}'\mathbf{P}_\theta\mathbf{A}^{-1}\mathbf{\Phi}\left(\mathbf{e} - \mathbf{e_{-z}}\right) \quad \textbf{(demand propagation)} \\ &+ \frac{1}{\theta}\mathbf{e_{-z}}'\mathbf{P}_\theta(\mathbf{A}^{-1} - \mathbf{A_{-z}}^{-1})\mathbf{\Phi}\mathbf{e_{-z}} \quad \textbf{(addional link effect)} \\ &+ \frac{1}{\theta}\mathbf{e_{-z}}'\left(\mathbf{P}_\theta - \mathbf{P}_{\theta,-\mathbf{z}}\right)\mathbf{A_{-z}}^{-1}\mathbf{\Phi}\mathbf{e_{-z}}. \quad \textbf{(price externality)}\end{aligned}$$

The demand influence coefficient is

$$\mathbf{v_b}' = \mathbf{e}'\mathbf{P}_\theta\mathbf{A^{-1}}\hat{\mathbf{P}}_\theta.$$

The supply influence coefficient is

$$\xi_\mathbf{b}' = \mathbf{e}'(\mathbf{B}')^{-1}.$$

Here we provide several useful lemmas.

Lemma 8. *We have*

$$\mathbf{E}\mathbf{1} = \mathbf{e}$$

and

$$\mathbf{E}\mathbf{e} = \mathbf{e}.$$

For vectors $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a}'\mathbf{b} \neq 1$*, it holds.*

$$\left(\mathbf{I} - \mathbf{a}\mathbf{b}'\right)^{-1} = \mathbf{I} + \frac{1}{1-\mathbf{a}'\mathbf{b}}\mathbf{a}\mathbf{b}'.$$

Proof. The first two results are trivial. The third result is a special case of Lemma 7 with $\mathbf{A} = \mathbf{I}$. $\square$

A3.2 A large buyer

In this case, $\mathbf{W} = a\mathbf{u}_1\mathbf{1}'_{-1}$. Lemma 8 implies

$$\mathbf{A}^{-1} = \mathbf{I} + a\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{e}_{-1}\mathbf{u}_1'$$

and

$$\mathbf{B}^{-1} = \mathbf{I} + ae_1\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{e}_{-1}\mathbf{u}_1'$$

The price vector is

$$\mathbf{p}_\theta' = \mathbf{w_0}'\mathbf{E}\left(\mathbf{I} + ae_1\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{e}_{-1}\mathbf{u}_1'\right)\left(\frac{\theta-1}{\theta}\right)^{\theta-1}.$$

The vector of individual profits is

$$\begin{aligned}\pi(\mathbf{e}) =& \frac{1}{\theta}\mathbf{P}_\theta\mathbf{A}^{-1}\mathbf{\Phi}\mathbf{e}\\ =& \frac{1}{\theta}\mathbf{P}_\theta\left(\mathbf{I} + a\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{e}_{-1}\mathbf{u}_1'\right)\mathbf{\Phi}\mathbf{e}\\ =& \frac{1}{\theta}\left(\mathbf{P}_\theta\mathbf{\Phi}\mathbf{e} + ae_1\phi_1\left(\frac{\theta-1}{\theta}\right)^{\theta}\mathbf{P}_\theta\mathbf{e}_{-1}\right).\end{aligned}$$

We now consider the decomposition. We first consider firm 1. The own profit is the first element of $\pi(\mathbf{e})$ and is

$$\frac{1}{\theta}\phi_1\left(w_{10} + a\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{w}_0'\mathbf{e}_{-1}\right)\left(\frac{\theta-1}{\theta}\right)^{\theta-1}.$$

The demand propagation part is

$$\begin{aligned}\frac{1}{\theta}\mathbf{e}_{b,-1}'\mathbf{P}_\theta\mathbf{A}^{-1}\mathbf{\Phi}(\mathbf{e}-\mathbf{e}_{-1}) =& \frac{1}{\theta}a\left(\frac{\theta-1}{\theta}\right)^{\theta}\phi_1\sum_{i=2}^{n}p_j^{-\theta-1}e_i\\ =& \frac{1}{\theta}a\left(\frac{\theta-1}{\theta}\right)^{\theta}\phi_1\sum_{i=2}^{n}w_{i0}\left(\frac{\theta-1}{\theta}\right)^{\theta-1}e_i.\\ =& \frac{1}{\theta}a\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\phi_1\mathbf{w}_0'\mathbf{e}_{-1}\end{aligned}$$

The network structure change part is

$$\begin{aligned}\frac{1}{\theta}\mathbf{e}'_{b,-1}\mathbf{P}_\theta(\mathbf{A}^{-1}-\mathbf{A}_{-1}^{-1})\mathbf{\Phi}\mathbf{e}_{-1} =&\frac{1}{\theta}\mathbf{e}'_{b,-1}\mathbf{P}_\theta a\left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{e}_{-1}\mathbf{u}'_1\mathbf{\Phi}\mathbf{e}_{-1}\\ =&0\end{aligned}$$

because $\mathbf{\Phi}$ is diagonal so $\mathbf{u}'_1\mathbf{\Phi}\mathbf{e}_{-1}=0$. The price change part is

$$\begin{aligned}&\frac{1}{\theta}\mathbf{e}'_{b,-1}(\mathbf{P}_\theta-\mathbf{P}_{\theta,-1})\mathbf{A}_{-1}^{-1}\mathbf{\Phi}\mathbf{e}_{-1}\\ =&\frac{1}{\theta}\mathbf{e}'_{b,-1}(\mathbf{P}_\theta-\mathbf{P}_{\theta,-1})\left(\mathbf{I}+a\left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{e}_{-1}\mathbf{u}'_1\right)\mathbf{\Phi}\mathbf{e}_{-1}\\ =&\frac{1}{\theta}(\mathbf{p}'_\theta-\mathbf{p}'_{\theta,-1})\mathbf{E}_{-1}\mathbf{\Phi}\mathbf{e}_{-1}\\ =&\frac{1}{\theta}\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\left(\mathbf{w_0}'\mathbf{E}\left(\mathbf{I}+a\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{e}_{-1}\mathbf{u}'_1\right)-\mathbf{w_0}'\mathbf{E}_{-1}\right)\mathbf{E}_{-1}\mathbf{\Phi}\mathbf{e}_{-1}=0.\end{aligned}$$

Next, we consider firm $z\geq 2$. The own profit is

$$\frac{1}{\theta}w_{z0}\left(\phi_z+ae_1\phi_1\left(\frac{\theta-1}{\theta}\right)^\theta\right)\left(\frac{\theta-1}{\theta}\right)^{\theta-1}.$$

The demand propagation part is

$$\frac{1}{\theta}\mathbf{e}'_{-z}\mathbf{P}_\theta\mathbf{A}^{-1}\mathbf{\Phi}(\mathbf{e}-\mathbf{e}_{-z})=0.$$

The network structure change part is

$$\begin{aligned}\frac{1}{\theta}\mathbf{e}'_{-z}\mathbf{P}_\theta(\mathbf{A}^{-1}-\mathbf{A}_{-z}^{-1})\mathbf{\Phi}\mathbf{e}_{-z} =&\frac{1}{\theta}\mathbf{e}'_{-z}\mathbf{P}_{\theta 1}a\left(\frac{\theta-1}{\theta}\right)^\theta(\mathbf{e}_1-\mathbf{e}_0)\mathbf{u}'_1\mathbf{\Phi}\mathbf{e}_{-z}\\ =&\frac{1}{\theta}\mathbf{e}'_{-z}\mathbf{P}_\theta a\left(\frac{\theta-1}{\theta}\right)^\theta(\mathbf{e}-\mathbf{e}_{-z})\phi_1 e_1=0.\end{aligned}$$

The price change part is

$$\begin{aligned}&\frac{1}{\theta}\mathbf{e}'_{-z}(\mathbf{P}_\theta-\mathbf{P}_{\theta,-z})\mathbf{A}_{-z}^{-1}\mathbf{\Phi}\mathbf{e}_{-z}\\ =&\frac{1}{\theta}(\mathbf{p}_\theta-\mathbf{p}_{\theta,-z})'\mathbf{E}_{-z}\left(\mathbf{\Phi}\mathbf{e}_{-z}+a\phi_1 e_1\left(\frac{\theta-1}{\theta}\right)^\theta\mathbf{e}_{-1,-z}\right)\\ =&\frac{1}{\theta}\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\left(\frac{\theta-1}{\theta}\right)^{\theta-1}a\phi_1 e_1 w_{z0}.\end{aligned}$$

The influence coefficients are

$$\begin{aligned}
\mathbf{v_b}' =&\mathbf{e}'\mathbf{P}_\theta \left(\mathbf{I} + a\left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{e}_{-1}\mathbf{u}_1'\right)\hat{\mathbf{P}}_\theta \\
=&\mathbf{e}' + p_1^{\theta-1} a \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{p}_\theta' \mathbf{e}_{-1}\mathbf{u}_1' \\
=&\mathbf{e}' + p_1^{\theta-1} a \left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{w_0}'\mathbf{e}_{-1}\left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{u}_1'.
\end{aligned}$$

Thus, $v_z = e_z$ for $z \neq 1$. As long as $e_1 = 1$, $v_1 > 1$. Also,

$$\boldsymbol{\xi_b} = \mathbf{e}'(\mathbf{B}^{-1})' = \mathbf{e} + a e_1 \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{e}_{-1}.$$

Thus, $\xi_1 = e_1$. For any $z \neq 1$ with $e_z = 1$, $\xi_z > 1$.

Online Appendix 4 Numerical Example

This section presents the results of numerical simulations. The first subsection illustrates the network-motivated forbearance lending in the two-bank case. The second subsection examines how demand and supply coefficients are related to each component of the externality, and how the elements of $\mathbf{W}$ affect them.

A4.1 Numerical example of the baseline model of two banks

To illustrate the results from our model so far, we simulate the model with randomly selected network structure $\mathbf{W}$ and $\mathbf{w_0}$, based on the parameter values found in available data. To construct a random $\mathbf{W}$, we make an adjacency matrix of an unweighted directed network with the scale-free property by the method proposed by Goh et al. (2001).[1]

The power law exponent of the degree distribution of in-degree, i.e., number of suppliers, and out-degree, i.e., number of customers, is set at 2.3, following the finding by Saito et al. (2007) from the supply-chain network in Japan in 2006. If an element w_{ij} is assigned with 1 in this process, we replace it with a random draw from the uniform distribution between 0 and 1. The diagonal elements are replaced with 0. To ensure that the sum of each row of $\mathbf{W}$ does not exceed 1, we divide each element by the row sum if the row sum is greater than 1. To avoid an extremely high column sum, we also divide each element by the column sum if the column sum is greater than 1. The resulting $\mathbf{W}$ is a random network satisfying the assumptions in our theory. Each element in $\mathbf{w_0}$ is also a random draw from the uniform distribution between 0.1 and 0.9. ϕ_i is a random draw from the uniform distribution between 0 and 1. θ is set at 4, which is consistent with the average

[1] We used the function `sample_fitness_pl` in the `igraph` library of R, with the option to remove the finite size effect. This function generates the adjacency matrix of a scale-free network whose degree density function is proportional to $d^{-\gamma}$, where d is degree and γ is the power law exponent greater than 2, from given values of the exponent γ, the number of nodes, and the number of edges.

mark-up in Japanese firms.[2] We assume that the number of edges in each network is four times the number of firms.[3]

We randomly assign each firm in this network to one of two banks, bank 1 or bank 2 (panel (a) in Figure A.1). We assume that the required amount of loan F_i is 0.04 and the half of it, 0.02, is the refinancing part for all firms.

In the first stage, we identify firms who are free from the debt overhang problem and able to obtain a loan at the competitive rate from any lenders by iteratively applying the mapping (20).

$$\bar{\mathbf{e}} = \mathcal{G}(\mathbf{e}),$$

where the i-th element of $\bar{\mathbf{e}}$ $(n \times 1)$ is given by

$$\bar{e}_i = \mathbb{1}[\pi_i(\mathbf{e}) \geq F_i].$$

We obtain the fixed point after the iteration of this mapping. We denote it $\mathbf{e}^\dagger$. An example result of this first-stage screening is illustrated in Panel (a) in Figure (A.1). The white nodes indicate firms who can obtain a loan at the competitive rate from any lenders. The black nodes indicate the firms for whom only Bank 1 is willing to extend a required amount of loan due to the debt-overhang problem. The gray nodes indicate firms for whom only Bank 2 is willing to lend.

In the second stage, we obtain a Nash equilibrium of lending decisions of these banks. We obtain the Nash equilibrium with the maximal number of surviving firms as the maximal fixed point of the round-robin algorithm, proposed by Topkis (1998, Chapter 4.3.1, pp. 185-190). Each player searches for the best response in turn, given the other's response in the previous turn.

The best response for bank b is the fixed point of the following mapping:

$$\hat{\mathbf{e}}_{\mathbf{b}} = \mathcal{H}(\mathbf{e}_{\mathbf{b}}; \mathbf{e}_{\mathbf{b'}}), \tag{A.4}$$

where the i-th element $(i \in D_b^c)$ of $\hat{\mathbf{e}}_{\mathbf{b}}$ $(n \times 1)$ is given by

$$e_i = \mathbb{1}[\Pi_b(\mathbf{e}_{\mathbf{b}}; \mathbf{e}_{\mathbf{b'}}) - \Pi_b(\mathbf{e}_{\mathbf{b},-\mathbf{i}}; \mathbf{e}_{\mathbf{b'}}) \geq F_i].$$

Algorithm 1 (Round-robin algorithm to find a Nash equilibrium)**.** *1. We can find the maximal Nash equilibrium by the following steps:*

(a) Set the initial value: $\mathbf{e}_{\mathbf{b}}^{\mathbf{0}}$ *and* $\mathbf{e}_{\mathbf{b'}}^{\mathbf{0}}$ *such that* $e_i = 1 \; \forall \; i \in D^c$*.*

(b) Compute the best response $\hat{\mathbf{e}}_{\mathbf{b}}^{\mathbf{1}}$ *for bank* b *under the given* $\mathbf{e}_{\mathbf{b'}}^{\mathbf{0}}$ *as the fixed point of the*

[2] The average mark-up (sales/cost of goods sold) across all sizes of corporations excluding the financial sector from 2007 to 2016 is 1.31 in the Financial Statements Statistics of Corporations by Industry, Japanese Ministry of Finance. Because the mark-up in our model is $\theta/(\theta-1)$, the corresponding θ is about 4.

[3] We find 3,457,644 links among 766,327 firms in 2011, in the corporate relation database, provided by Tokyo Shoko Research. The average number of links per firm is about 4.5.

mapping (A.4),

$$\hat{\mathbf{e}}_{\mathbf{b}}^{\mathbf{1}} = \mathcal{H}(\hat{\mathbf{e}}_{\mathbf{b}}^{\mathbf{1}}; \mathbf{e}_{\mathbf{b'}}^{\mathbf{0}}).$$

(c) *Compute the best response* $\hat{\mathbf{e}}_{\mathbf{b'}}^{\mathbf{1}}$ *for bank* b' *under the given* $\hat{\mathbf{e}}_{\mathbf{b}}^{\mathbf{1}}$ *as the fixed point of the mapping (A.4),*

$$\hat{\mathbf{e}}_{\mathbf{b'}}^{\mathbf{1}} = \mathcal{H}(\hat{\mathbf{e}}_{\mathbf{b'}}^{\mathbf{1}}; \hat{\mathbf{e}}_{\mathbf{b}}^{\mathbf{1}}).$$

(d) *Repeat the procedures (b) and (c) until each of* $\hat{\mathbf{e}}_{\mathbf{b}}$ *and* $\hat{\mathbf{e}}_{\mathbf{b'}}$ *converges to a certain value. The convergence will be achieved by the assured existence of a fixed point for the mapping.*

2. *To find the minimal Nash equilibrium, we start the above algorithm from the initial value of* $\mathbf{e}_{\mathbf{b}}^{\mathbf{0}} = \mathbf{e}_{\mathbf{b'}}^{\mathbf{0}} = (0, \cdots, 0)'$.

Panels (b), (c), and (d) illustrate the Nash equilibrium after the round-robin algorithm. The black nodes in these panels are those who are financially distressed, i.e., $\pi_i(\mathbf{e}) < F_i$, and cannot obtain a forbearance lending (firms 8, 10, 13, and 23). The blue nodes are those who are financially distressed but can obtain a forbearance lending (firms)

The size of each node indicates the relative value of the demand influence coefficient v_i (Panel (a)), the price influence coefficient ξ_i (Panel (b)), and the total externality, i.e., the sum of (26)-(28) (Panel (c)). These values for those distressed firms are listed in Table A.1. The table indicates that those with higher influence coefficients and higher externality tend to obtain forbearance lending.

We emphasize that those who obtain a forbearance lending are so-called "zombie firms" under the criteria by the existing studies, i.e., to keep borrowing at the interest rate lower than the prime rate (Caballero et al., 2008). Thus, the estimation based on the existing criteria for zombie firms may overestimate the problem of welfare-deteriorating forbearance lending.

Table A.1: Values of distressed firms in the two-bank case (evaluated before the round-robin algorithm)
(Notes) The values for distressed firms in Panels (b)-(d) of Figure A.1 are listed.

(a) borrowers of Bank 1

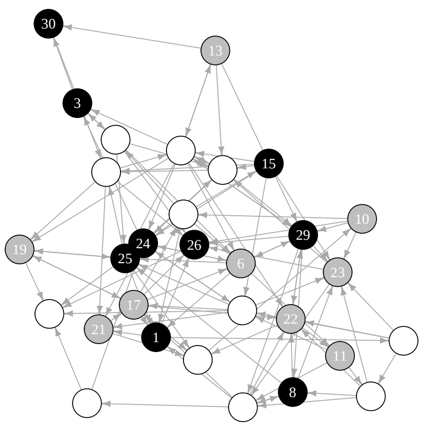
ID	Externality	Demand propagation	Additional link	Price externality	v_i	ξ_i	Forbear
1	0.007	0.001	0.001	0.005	1.068	1.120	YES
3	0.013	0.002	0.001	0.011	1.109	1.509	YES
8	0.002	0.000	0.000	0.002	1.023	1.140	NO
15	0.003	0.000	0.000	0.003	1.001	1.132	NO
24	0.008	0.003	0.001	0.004	1.224	1.200	YES
25	0.009	0.003	0.002	0.004	1.201	1.221	YES
26	0.010	0.006	0.001	0.003	1.401	1.085	YES
29	0.008	0.005	0.001	0.002	1.143	1.146	YES
30	0.002	0.001	0.000	0.000	1.043	1.007	YES

(b) borrowers of Bank 2

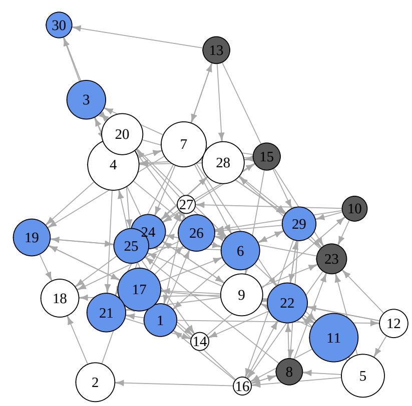
ID	Externality	Demand propagation	Additional link	Price externality	v_i	ξ_i	Forbear
6	0.013	0.001	0.001	0.011	1.067	1.241	YES
10	0.001	0.000	0.000	0.001	1.014	1.113	NO
11	0.028	0.001	0.003	0.025	1.143	1.432	YES
13	0.002	0.001	0.000	0.001	1.204	1.067	NO
17	0.019	0.007	0.004	0.008	1.378	1.225	YES
19	0.011	0.007	0.001	0.002	1.235	1.119	YES
21	0.013	0.001	0.001	0.010	1.088	1.275	YES
22	0.015	0.010	0.004	0.001	1.413	1.095	YES
23	0.004	0.000	0.001	0.003	1.153	1.055	NO

Figure A.1: Fixed point in the two-bank case

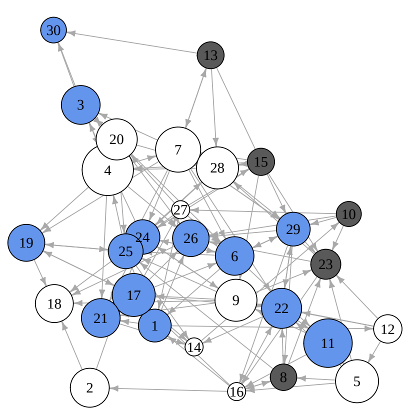
(Notes) An example network of $n = 30$. Arrows indicate the direction of product sales. The number of edges is 120. The power law exponent of in-degree and out-degree is 2.3. The network is generated by the procedure described in Section A4.1. The gray and blue nodes are distressed firms by the criterion $\pi_i(\mathbf{e}) < F_i$. The blue ones are those that obtain forbearance lending. $F_i = 0.04$ and $f_i = 0.02$ for all firms. Panel (a) indicates the firms who can obtain a loan from any banks at a competitive rate (white nodes) and those who can obtain a loan only from Bank 1 (black nodes) or Bank 2 (gray nodes). Panels (b), (c), and (d), indicate the fixed point of the round-robin algorithm. The size of each node in these panels indicates the relative value of (b) demand influence coefficient, (c) price influence coefficient, and (d) total externality, of each firm evaluated at the initial stage of the round-robin algorithm. The black nodes in these panels are those who are distressed and cannot obtain a forbearance loan. The blue ones are those who are distressed but obtain forbearance lending. The node IDs are consistent over all panels.



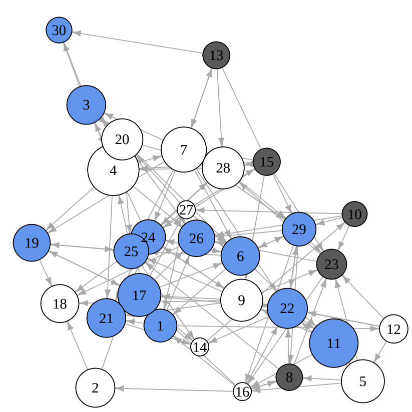
(a) bank assignment



(b) Demand influence coefficient v_i



(c) Price influence coefficient ξ_i



(d) Total externality (28)

Table A.2: Percentile of simulated correlation coefficients between row/column sum of **W** and externality/influence coefficient
(Notes) Each cell reports the percentiles of correlation coefficients calculated in each of 1,000 simulated networks of 30 firms. Each network is constructed by the procedure described in Section A4.1.

	(i) row sum of **W**					(ii) column sum of **W**				
	1%	10%	med	90%	99%	1%	10%	med	90%	99%
demand propagation	0.28	0.41	0.56	0.62	0.68	-0.45	-0.25	0.00	0.13	0.24
additional link effect	0.42	0.54	0.67	0.73	0.77	0.14	0.31	0.48	0.56	0.62
price externality	-0.21	-0.02	0.23	0.34	0.45	0.28	0.44	0.60	0.67	0.72
externality	-0.23	-0.01	0.25	0.35	0.44	-0.12	0.10	0.32	0.42	0.50
v_i	0.30	0.52	0.66	0.72	0.76	-0.45	-0.24	-0.01	0.12	0.25
ξ_i	-0.39	-0.25	-0.02	0.12	0.25	0.97	0.98	0.98	0.99	0.99

A4.2 W and the externality for each firm by simulation

In this section, to complement the analytical comparative statics in Section 3.3, we numerically show the relationship between each component of the externality, the influence coefficients, and the row and column sums of **W**. Hereafter, we assume that all the firms face the debt-overhang and the monopolistic bank, but the result remains qualitatively the same in the baseline case where some firms are free from the debt-overhang problem.

Toward this goal, we construct a simulated data sample by repeating the random simulation described in Section A4.1, 1,000 times under the same value of parameters. We report the set of correlation coefficients between key variables in this simulated data.

To see the pattern of the correlation between the sum of the i-th row of **W** and the externalities and influence coefficients of firm i in each simulated network, we calculate these correlation coefficients using 30 firms in each of 1,000 simulated networks. Table A.2 lists the percentiles of the 1,000 correlation coefficients.

Each percentile in Column (i) in the table shows that the row sum of **W** is highly positively correlated with the demand-propagation effect (26) and the demand influence coefficient v_i. The row sum is also positively correlated with the additional link effect (27).

Each percentile in Column (ii) in the table shows that the column sum of **W** is highly correlated with the price externality, and the price influence coefficient ξ_i. The column sum is also positively correlated with the additional link effect (27).

Column (i) of Table A.3 lists the percentiles of correlation coefficients of the demand influence coefficient v_i and each component of the externality calculated in each of 1,000 simulated networks. All percentiles indicate that the demand influence coefficient is highly positively correlated with the demand propagation effect and the additional link effect, whereas it tends to be negatively correlated with the price externality. v_i is positively correlated with the total externality in more

Table A.3: Percentiles of simulated correlation coefficients between influence coefficients and externality
(Notes) Each cell reports the percentiles of correlation coefficients calculated in each of 1,000 simulated networks of 30 firms. Each network is constructed by the procedure described in Section A4.1.

	(i) v_i					(ii) ξ_i				
	1%	10%	med	90%	99%	1%	10%	med	90%	99%
demand propagation	0.04	0.22	0.42	0.53	0.61	-0.45	-0.25	-0.01	0.12	0.24
additional link effect	0.14	0.32	0.51	0.60	0.67	0.13	0.30	0.49	0.56	0.62
price externality	-0.57	-0.43	-0.24	-0.14	-0.02	0.29	0.45	0.62	0.69	0.74
externality	-0.03	0.15	0.38	0.49	0.58	-0.10	0.12	0.33	0.43	0.52
ξ_i	-0.45	-0.25	-0.01	0.12	0.25					

than 90% of simulated networks.

Column (ii) of Table A.3 lists the percentiles of correlation coefficients of the demand influence coefficient ξ_i and each component of the externality calculated in each of 1,000 simulated networks. The price influence coefficient ξ_i is highly correlated with the price externality, and the additional link effect to a lesser extent. It is positively correlated with the total externality in more than 90% of simulated networks.

To examine whether firms with higher demand or price influence coefficients tend to be a target for forbearance lending, we fit the forbearance probability by a linear probability model with the simulated sample of distressed firms. The dependent variable is the dummy variable indicating that a firm obtained forbearance lending after the first screening by the condition (22). The independent variables include the demand influence coefficient v_i and the price influence coefficient ξ_i. We control for the fixed effect for each of 1,000 simulated networks. The simulated sample is constructed under the identical refinancing cost of $F_i = 0.03$.

The result with 5,039 firms with negative individual profit in the simulation is

$$\text{Prob(obtain forbearance lending)} = 0.433 + 0.302 v_i + 0.874 \xi_i. \qquad (\text{Adj.}R^2 = 0.122).$$

The estimated coefficients show firms with higher influence coefficients obtain forbearance lending.

A4.3 Number of edges per firm

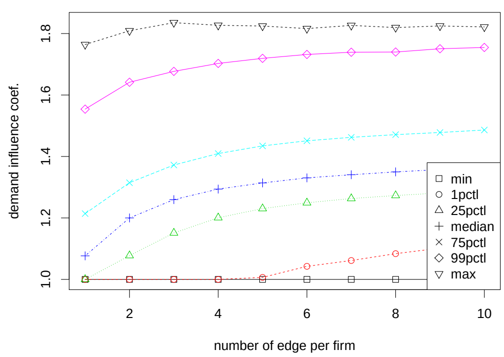
We examine the effect of the number of edges per firm among typical network features. The larger the number of edges per firm, the more densely connected the network. This could increase the influence of each firm on other firms because the reach of the externality expands. To examine this point, we simulated networks of 30 firms 1,000 times under the parametric assumptions, described in Section A4.1, with various numbers of edges per firms from one to 10.

The number of edges, or the network density, has a significant positive impact on both influence coefficients at most percentiles (panels (a) and (b)). The dispersion of the demand influence coefficient between the 1st and 99th percentile remains wider than that of the price influence coefficient, as the number of edges increases.

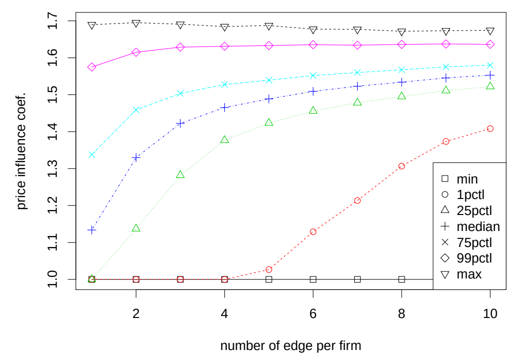
Under our parameters, the price externality accounts for the largest part of the externality (panel (e)). The demand propagation effect is the second largest (panel (c)). The total externality per profit (panel (f)) implies that, for example, when the number of edges per firm is four, a 99th percentile firm can obtain forbearance lending as long as the required additional loan amount f_i is less than about three times its profit (note that the condition (22) implies that a bank is willing to lend if the sum of the externality and the profit is greater than the refinancing cost).

All the components of the externality at most percentiles are increasing in the number of edges per firm. This result provides reassurance that a more densely connected network entails a positive externality among firms. It also implies that forbearance lending emerges more often in a more densely connected network.

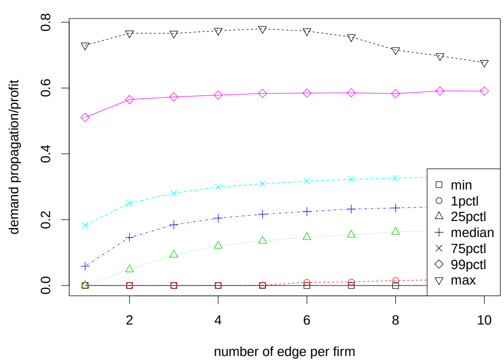
Figure A.2: Number of edges per firm and externality distribution in simulation
(Notes) Horizontal axis indicates the number of edges per firm. Vertical axis indicates the percentiles of influence coefficients or externality per profit, which is calculated from the result of 1,000 simulations of a random network. Each network is constructed by the procedure and parameters described in Section A4.1. The number of edges per firm takes values from one to 10.



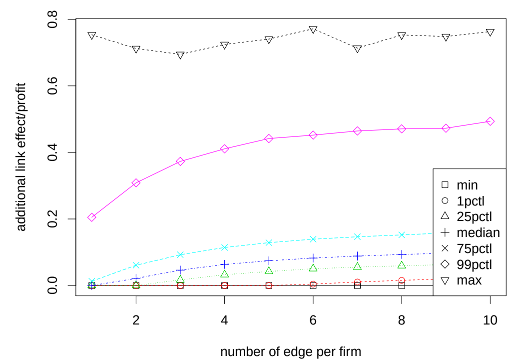
(a) Demand influence coefficient v_i



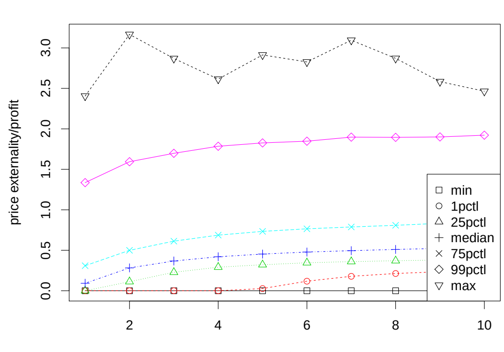
(b) Price influence coefficient ξ_i



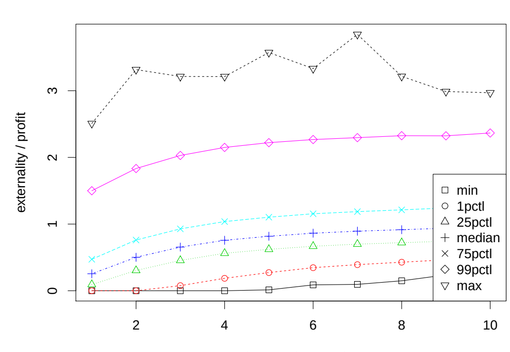
(c) Demand propagation effect per profit



(d) Additional link effect per profit



(e) Price externality per profit



(f) Total externality per profit

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