Network-motivated Forbearance Lending*

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Abstract

This paper develops a theoretical framework for network-motivated forbearance lending, or forbearance lending to influential buyers and sellers in a supply network. Because dominant banks in a financial market internalize the negative externality of an influential firm's exit, they may continue to refinance a loss-making influential firm at an interest rate lower than the prime rate. This type of forbearance lending is distinct from other strategies such as evergreening or gambling for resurrection, and we show that network-motivated forbearance lending is independent of the financial soundness of the bank and can be welfare improving. To evaluate the extent of the externality of an influential seller or buyer, we propose two measures: the price influence coefficient and the demand influence coefficient, respectively.
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JEL Classification: C55, D57, G21, G32, L13, L14

1 Introduction

In economic downturns, we frequently observe financial institutions engaging in *forbearance lending*; that is, the repeated refinancing of existing loans to underperforming or nonperforming firms at extremely low interest rates. Examples in the literature include major Japanese banks during the banking crisis in the late 1990s (e.g., Sekine et al., 2003; Peek and Rosengren, 2005; Caballero et al., 2008) and European banks during the 2012 debt crisis (Acharya et al., 2019). A robust understanding of the mechanisms of forbearance lending is important because it helps us develop a conceptual framework for assessing the associated economic costs and benefits. Existing theories have focused on forbearance lending within a one-to-one relationship between bank and firm, with *relational banking theory* considering overinvestment in monitoring to be an underlying mechanism of forbearance lending in this context (e.g., Dewatripont and Maskin, 1995; Berglöf and Roland, 1997). Meanwhile, other concepts apply only to undercapitalized banks, with the *evergreening hypothesis* (Peek and Rosengren, 2005) focusing on an undercapitalized bank's incentive to conceal a loan loss from a nonperforming firm and the *gambling-for-resurrection hypothesis* (e.g., Bruche and Llobet, 2014) highlighting its incentive to make a risky bet on the resurrection of a nonperforming firm. An overarching implication in the theoretical discussion of forbearance lending is that it is a suboptimal and welfare-destructive lending practice.

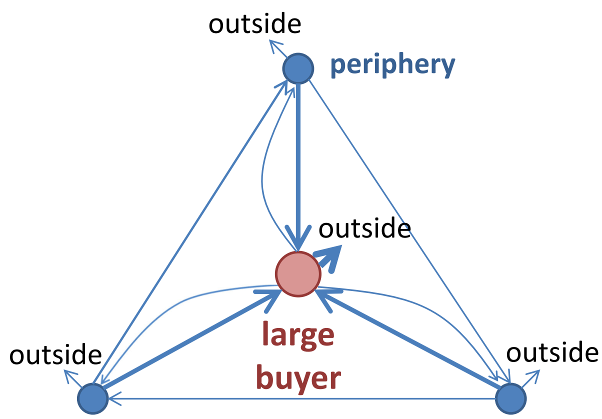
While these existing theories capture important aspects of a bank's motivation for forbearance lending, the latter two do not account for this practice by financially sound banks, and all three overlook the fact that a bank lends not merely to one firm but to thousands of firms and can also observe their interactions. Even the financial media has noted that banks recognize the potential spillover effects of forbearance lending, highlighting the importance of considering industry-level interactions.[1] Moreover, the fact that credit information vendors such as Bloomberg and Teikoku Data Bank have collected information on the supply network further indicates the usefulness of such information to potential creditors. The present study attempts to deepen our understanding of forbearance lending by theoretically demonstrating another generating mechanism that has been overlooked by existing theories.

More specifically, we present a model showing that profit-maximizing banks may also undertake forbearance lending in order to sustain the supply network among borrowers, which we call *network-motivated forbearance lending*. Unlike the evergreening or gambling hypotheses, this type of lending occurs regardless of the financial soundness of the bank. Furthermore, it may improve welfare, which sharply contrasts with existing theories that conclude that forbearance lending is harmful.

To illustrate the economic problem addressed in this paper, let us consider the example shown in Figure 1. The firm at the center is a large buyer of intermediate goods produced by the peripheral firms around it. The direction of each arrow in the figure indicates the flow of products, and its size

[1] For example, an article in the Nikkei newspaper reported: "[A regional] bank is carefully avoiding hasty bad-loan write-offs, which can be devastating for the local economy, its revenue base. In navigating the turnaround of distressed borrowers, it gives priority to large borrowers with a larger number of customers and suppliers in order to avoid a chain of bankruptcies." "Rettou Kin'yu Fairu Tochigi Hatsu," (Archipelago Finance File from Tochigi) Nikkei Finance Newspaper, p. 3, February 28, 2006. Translation by the authors.

Figure 1: A directed and weighted supply network



Note: Each node is a hub or a peripheral firm. The direction of each arrow represents the direction of product sales, and its size indicates sales volume.

represents sales volume. Given the size of the large buyer at the center, the negative impact if it were to close would propagate throughout the network by reducing the sales of the peripheral firms and could even trigger a chain of closures.[2] If the total profit of the network is positive despite the losses incurred by the large buyer, it is more profitable to keep it afloat.

Profit-maximizing banks are willing to provide forbearance lending if they can exercise dominant power over firms in the network because they can then expect larger profits by supporting a loss-making influential firm and recouping that cost through higher interest rates imposed on the peripheral firms. This paper formally investigates this phenomenon and identifies various avenues through which a well-connected firm may contribute to a bank's profit beyond its own individual profit.

While network-motivated forbearance is not exclusively provided by banks,[3] a typical provider is a bank that is a dominant lender in a regional economy that has market power, information, and exposure obtained through long-term lending relationships with local firms. This allows the bank to internalize negative externalities and exert market power to engage in network-motivated forbearance to overcome debt overhang problems within the network. As banks are often dominant lenders in a regional economy, they have existing loans to firms in the regional supply network. Firms with a large existing senior debt are often unable to obtain financing in the competitive market to develop new projects because of its debt overhang (Myers, 1977), with only the bank that provided the senior loan being willing to provide additional financing to this firm because it

[2]While Figure 1 illustrates the case of a large buyer, a similar logic applies to the case of a large seller that produces input goods for peripheral firms around it. The closure of the large seller would increase the production costs of peripheral firms and reduce their profits by damaging their competitive edge in the product market.

[3]While our primary focus in this paper is on the behavior of banks, the core predictions of our theory are broadly applicable to any entities such as governments or large hedge funds possessing information, exposure, and bargaining power within a supply chain network.

might improve the chances for repayment of the senior loan. In this manner, the bank holding senior loans can exercise market power over firms facing a debt-overhang problem, which enables the bank to engage in forbearance lending.

To formally analyze the mechanism behind network-motivated forbearance lending, we construct a supply network model with a banking sector. The banking sector is an oligopoly, with banks making refinancing decisions for each firm. Moreover, due to debt overhang, each bank may exercise market power over certain firms in the supply network. We formulate an algorithm to obtain a loan portfolio that satisfies the necessary conditions for bank profit maximization. These conditions are closely related to the positive externalities of one firm to other firms connected through a supply chain network. As externalities are not fully internalized in a differentiated product market where each firm sets its price simultaneously, a bank with existing long-standing lending relationships with mutually connected firms can detect such externalities and is incentivized to undertake forbearance lending to internalize them.

The model enables us to decompose the positive externalities into three components: 1) *demand propagation*, 2) *price externality*, and 3) *additional-link effect*. The first component captures the demand spillover from a firm to its suppliers, while the second component is the spillover of cost efficiency for connected firms due to an additional variety of inputs. The third component captures the positive contribution of a firm through increased network connectivity and reinforcement of the former two components. This decomposition leads to a clear understanding of the cause of the externality. In particular, it provides clear guidance regarding which features of firms are important in determining the possibility of obtaining forbearance lending.

Additionally, this decomposition provides us with a way of constructing empirical measures of each component of an externality; specifically, the first two components, which are more interesting and relevant for our purposes. We propose a measure of the extent of the demand propagation of each firm, which we call the *demand influence coefficient*, and of the price externality of each firm, which we call the *price (supply) influence coefficient*. These metrics are useful for future empirical studies and policy evaluations of forbearance lending.

Our theory suggests several testable hypotheses and policy implications. First, a firm with a high influence coefficient in terms of demand and supply is more likely to be the target of network-motivated forbearance lending; that is, it can obtain a loan at a rate lower than the prime rate despite poor performance. Second, network-motivated forbearance should be observed more often when there is a large or dominant bank. Third, forbearance lending imposes higher interest costs on peripheral firms in the network. As for policy implications, an important implication is that forbearance lending is not always welfare destructive. In fact, our theory suggests that financially sound banks may offer forbearance lending to keep the supply network intact, which could lead to welfare improvement.

In practice, we can find several examples of bank-coordinated or government-led bailouts that consider the externality of a firm. One example of a bailout focusing on demand propagation is the US government bailout of the Big Three automakers that provided about 8 billion USD during the

2008 global financial crisis despite heated controversy over this action. A news article reported that, "The Big Three directly employ almost 250,000, $[\cdots]$, not counting the vast network of suppliers and dealers whose businesses are intertwined. In all, administration officials estimate that the failure of the US automakers would cost the economy more than one million jobs."[4]

A historical example of a credit allocation policy focusing on price externality is the cooperative arrangement of the Japanese Bankers Association in 1951 under the guidance of the Ministry of Finance, which urged banks to allocate funds primarily for key intermediate goods sectors such as electric power, maritime transportation, coal mining, and steel, which were primary inputs for all industries. The purpose was to accelerate the rebuilding of the productive capacity of the economy and simultaneously tame inflation pressure (pp. 7–11, Okazaki et al., 2002). Among these sectors, maritime transportation, which suffered severe financial distress due to the loss of 82% of its ships during wartime, received subsidies for interest payments from the government as well as postponement of interest payments to banks (pp. 248–255, Okazaki et al., 2002).

Related literature. The novelty of network-motivated forbearance within the existing literature, which is based on a one-to-one relationship between a bank and a firm, is that it focuses on the fact that a bank lends to thousands of firms interconnected through a supply network. An additional novel theoretical point is that a bank may have an incentive to provide network-motivated forbearance regardless of its financial soundness, and this forbearance may be welfare enhancing.

The theoretical literature on forbearance proposes policies to mitigate the propagation of a shock among mutually financing banks in a financial network. Demange (2018), for example, shows that a liquidity injection to the bank with the largest threat index, a measure of the spillover effect through connected banks, is an effective remedy against such a shock. Further, the literature has shown that connected banks are incentivized to provide autonomous bailouts of a troubled adjacent bank either by liquidity transfer (Leitner, 2005) or by organizing a consortium (Rogers and Veraart, 2013). A novel point of our study is that we extend the existing literature focusing on a financial network to a supply chain network instead. We characterize forbearance lending as a strategic response by a bank to mitigate shock propagation throughout the network by preferentially supporting influential firms with larger externalities. This externality in a supply network (i.e., demand propagation and price externality) is qualitatively different from an externality through a chain liquidity shortage in a financial network. Our framework also enables us to consider not only directly connected firms but also indirectly connected firms and, further, to propose measures of the extent of the spillover of each firm; namely, demand and price influence coefficients. While we use these coefficients to indicate the likelihood of forbearance, they can also be viewed as threat indices of demand and price spillover analogous to Demange (2018).

As for the empirical literature, many studies report significant propagation effects of negative shocks through an interfirm transaction network (Hertzel et al., 2008; Boissay and Gropp, 2013; Chen et al., 2013; Barrot and Sauvagnat, 2016; Jacobson and von Schedvin, 2015; Azizpour et al.,

[4] "Detroit Gets Access To Bailout Funds," Dec. 13, 2008, *The Wall Street Journal.*

2018; Carvalho et al., 2021).[5] More broadly, several studies find evidence of a spillover effect from the bankruptcy or financial distress of firms in a geographical neighborhood (Bernstein et al., 2019) and within a sector (Giannetti and Saidi, 2019); however, the present paper focuses on the bank's response to this spillover and highlights the bank's motivation to provide forbearance lending to avoid negative externalities throughout the supply network. Some papers consider a similar mechanism from the different perspective of the negative spillover of fire sales to asset prices. For example, Favara and Giannetti (2017) show both theoretically and empirically that a bank with an extensive lending share in a specific region tends to refrain from foreclosures on mortgages to avoid the negative externality of fire sales of collateralized assets.[6] Similarly, Giannetti and Saidi (2019) find that, for the same reason, dominant banks are more willing to provide loans to a distressed sector for which the bank is a dominant lender. In addition, they also find that banks are willing to support suppliers and customers of the sector for which the bank is a dominant lender by providing more loans when they are distressed. Our study is a theoretical attempt to identify how lending behavior is affected by an externality within a supply network, and our goal is to analytically formulate the mechanism generating this lending behavior from the viewpoint of the externality of a firm to connected firms within the network.

In our theoretical model, we formulate the interfirm supply network as a directed and weighted network of sales among oligopolistic firms offering differentiated products that are both intermediate inputs and final products. This type of multisector model has already been proposed in macroeconomics to analyze the behavior of the aggregate economy (e.g., Long and Plosser, 1983; Dupor, 1999; Horvath, 2000; Acemoglu et al., 2012; Baqaee, 2018; Bigio and La'o, 2016, among others). We simplify the Baqaee (2018) model and add the financial sector, which strategically determines the credit allocation. The concepts of our demand and price influence coefficients are different from those of the influence vector of Acemoglu et al. (2012) and the demand and supply centrality proposed by Baqaee (2018). While those measures capture the vulnerability of each firm to a demand or productivity shock in the final goods sector, ours are designed to measure the extent of a firm's spillover to its connected firms.

An important concept related to our network-motivated forbearance is "zombie lending," introduced by Caballero et al. (2008), which is rescue lending for a nonperforming firm at an unnaturally low interest rate below the prime rate. This literature has suggested that evergreening (Peek and Rosengren, 2005; Faria-e-Castro et al., 2024) and gambling-for-resurrection (Bruche and Llobet, 2014) drive banks toward zombie lending and that both cause economically inefficient outcomes. By contrast, we show that the externality through a supply network is another motivation for forbearance lending, and this type is not necessarily detrimental to the economy.

[5] Regarding this point, Campello and Gao (2017) find that banks impose a higher interest rate on borrowers whose sales are concentrated on a certain customer as a precautionary response against firms with a less-diversified clientele.

[6] Additionally, Gupta (2022) presents a theoretical model where a dominant bank is incentivized to oversupply mortgages to prop up housing prices.

Organization of the paper. The remainder of this paper is organized as follows. In the next section, we present the model of a supply network and derive the equilibrium. This section contains our main theoretical results on network-motivated forbearance. In particular, we discuss the necessary conditions for network-motivated forbearance and characterize the sources of the underlying externality. Section 3 then introduces influence coefficients to measure the externality and characterize firms obtaining network-motivated forbearance, proposing several testable hypotheses. Section 4 concludes and discusses the limitations of our analysis. The proofs of key propositions are presented in the appendix. The proofs of other propositions and numerical examples are presented in the Online Appendix.

2 Theoretical model

This section presents a theoretical model of a supply network, including the banking sector, where two banks operate noncooperatively. We derive the equilibria and elucidate the necessary conditions for forbearance lending. Furthermore, we decompose the externalities that precipitate forbearance lending and compare network-motivated forbearance lending with other types of forbearance lending.

2.1 Setup

We model the products market as an oligopoly and the banking market as a duopoly. The production technology requires intermediate goods and creates a supply network. We also assume that firms must refinance their existing bank loans to keep operating, and the role of the banking sector in our model is to provide this refinancing.

The product market consists of a continuous mass of firms producing differentiated products and a continuous mass of price-taking consumers. Among them, our analysis focuses on a supply chain network consisting of a finite n firms that have existing loans needing to be refinanced by a bank. The firms in the network are indexed by i $(= 1, 2, \ldots, n)$, where firm i produces and prices good i, which can be a consumption good, an intermediate good, or both. The quantity and the price of good i are denoted by x_i and p_i, respectively.

The firms in the network and a bank refinancing them treat the prices of products supplied from outside the network as exogenous variables in their decisions. We assume that the basket of inputs from outside the network, which is indexed by $i = 0$, is identical across firms, and we normalize the price of the basket to one (i.e., $p_0 = 1$) to simplify the analysis. This outside product basket serves as the numéraire; that is, all values in our model are denominated in terms of this basket. However, this simplification does not substantially affect our main analysis.

We assume the following decision timing and settlement scheme.

Time 1. *Financial market:* Each firm has an existing loan that must be refinanced. The banks decide which firms will be refinanced competitively for firms free of debt overhang or monopolistically for firms with a debt overhang problem (described later). The banks also have an outside risk-free investment opportunity with a zero rate of return. It is common knowledge which firms have been refinanced, and firms that do not obtain refinancing exit the economy.

Time 2. *Product market*

(1) *Production and input demand schedule:* Firms decide their production schedule and input-demand schedule to minimize the cost for any given vector of goods prices. The product-demand schedule outside the network, such as that of consumers, is determined at the same time. The production function of each firm is common knowledge.

(2) *Price setting:* Each firm decides the price of its product (and quantity to be produced) in simultaneous moves to maximize each firm's profit under the demand schedule mentioned above. The prices of all products become common knowledge, and all payments for intermediate inputs are made by trade credit.

Time 3. *Outcome:* Firm profits and outside investment outcomes are realized. Firms settle all trade credits. The bank obtains returns from its loans to firms and the outside investment opportunity.

For firms with a debt overhang, only banks with existing exposure are willing to refinance. Such a bank behaves as a monopolistic lender if it is the only bank with existing exposure and can capture the entire profit of the firm, meaning that loans and equities are indistinguishable for this group of firms.

Let e_i equal one if firm i operates, or zero otherwise. Good i is not supplied if $e_i = 0$. We assume that $e_0 = 1$; that is, inputs supplied outside the network always exist.

2.2 Firms in a network

The n firms inside a network have the following CES production function:

$$x_i = \left(\sum_{j=0}^{n} w_{ij}^{\frac{1}{\theta}} x_{ij}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}, \quad \theta > 1, \quad i = 1, 2, \cdots, n,$$

where θ is the elasticity of input substitution, w_{ij} is the technological importance of input j for the production of firm i, and x_{ij} is the quantity of inputs supplied by firm j used in production by firm i. x_{i0} represents inputs outside a network, which may include labor.

There are two assumptions underlying this function. First, input goods are imperfectly substitutable, so each producer has a certain level of market power. Second, the function exhibits the love-of-variety property so that productivity increases as the variety of inputs increases. Our proofs of the nonnegative externality of a firm and the super-modularity of the lending game between two banks, which are discussed in later sections, depend crucially on this property.

The sequence of technological importance, $\{\{w_{ij}\}_{i=1}^{n}\}_{j=0}^{n}$, is the most important determinant of the supply network. We assume that the supply network is rigid, namely, that w_{ij} does not change even if a supplier of an intermediate product exits.[7] This assumption also means there is no free entry of new firms into the market.

We assume that $\{\{w_{ij}\}_{i=1}^n\}_{j=0}^n$ satisfies $0 \leq w_{ij} \leq 1$, $\forall\ i, j$; $w_{ii} = 0$; and $0 < \sum_{j=1}^n w_{ij} \leq 1$.[8] Note that because some firms may fail to operate, the actual supply network becomes $\{\{e_j w_{ij}\}_{i=1}^n\}_{j=0}^n$.

Each firm must refinance its existing loan of F_i from a bank to continue its operations. A firm that can obtain refinancing always operates even when it results in a loss because the bank bears its losses.

2.2.1 Demand for intermediate products

The intermediate demand function for firm i for each product at any production level is derived by solving the following cost-minimization problem:[9]

$$\min_{\{x_{ij}\}_{j=0}^n} \sum_{j=0}^n p_j x_{ij}, \quad \text{s.t., } x_i = \left(\sum_{j=0}^n w_{ij}^{\frac{1}{\theta}} x_{ij}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \quad \text{and } (1-e_j)x_{ij} = 0,$$

where the second constraint is the availability constraint. The usual cost minimization gives the demand of firm i for good j:

$$x_{ij} = \left(\frac{p^i}{p_j}\right)^{\theta} e_j w_{ij} x_i, \text{ where } p^i \equiv \left(\sum_{j=0}^n e_j w_{ij} p_j^{-(\theta-1)}\right)^{-\frac{1}{\theta-1}}, \tag{1}$$

for $i = 1, 2, \cdots, n$ and $j = 0, 1, \cdots, n$. p^i is the average cost of producing good i because

$$\sum_{j=0}^n e_j p_j x_{ij} = p^i x_i. \tag{2}$$

The total intermediate demand for good j is

$$\sum_{i=1}^n e_i x_{ij} = \sum_{i=1}^n \left(\frac{p^i}{p_j}\right)^{\theta} e_i e_j w_{ij} x_i. \tag{3}$$

2.2.2 Demand from outside the network

There are two types of demand for good i from outside the network: firms outside the network demand good i as an input, and consumers also demand it as a final consumption good. We assume

[7] This assumption is more plausible in industries where the designs of input products or the contents of services are highly customized, information- or skill-intensive, and specific to each user. Automobiles, construction, and some types of retailers/wholesalers dealing in custom-made items are of this type.

[8] The value of $\sum_{j=0}^n w_{ij}$ represents the productivity level. Note also that the condition $\sum_{j=1}^n w_{ij} \leq 1$ guarantees the existence of the equilibrium price vector (10) as well as the existence of an equilibrium.

[9] To keep the exposition simpler, we assume that each firm behaves as a price-taker when it purchases inputs. A monopsony case is possible, but it complicates the analysis substantially.

that h_i, the total demand from outside the network for good i, is proportional to $p_i^{-\theta}$:

$$h_i = \phi_i p_i^{-\theta}, \tag{4}$$

where ϕ_i is a nonnegative constant. The foundations for this assumption are provided in Online Appendix 2 and are similar to the calculation in Section 2.2.1.

2.3 Financial market

Each firm must obtain a loan F_i to continue operating. The required loan consists of two parts: $F_i = l_{ib} + f_i$, where the first part captures the amount for refinancing an existing senior loan l_{ib} from bank b, and the second part is the new funding required to keep operating, f_i, which is junior to the first part. We assume that both l_{ib} and f_i are nonnegative. We also assume that the loanable funds of each bank exceed the total demand for funds.

Any potential lender is willing to provide a junior loan f_i if

$$\pi_i(\mathbf{e}) - l_{ib} \geq f_i, \tag{5}$$

where $\mathbf{e}$ is the $n \times 1$ vector of which the i-th element is e_i and $\pi_i(\mathbf{e})$ is the profit of firm i under $\mathbf{e}$. The set of surviving firms, $\mathbf{e}$, and the profit $\pi_i(\mathbf{e})$ are determined as the outcomes of the noncooperative game in which each potential lender makes lending decisions based on (5). The left-hand side is the maximal amount that can be assumed for the repayment of the junior loan after repaying the senior loan of l_{ib}, and the right-hand side is the principal of the junior loan. This condition implies that the principal and the opportunity cost of the junior loan can be fully repaid.

If the firm does not meet this condition, only the incumbent lender, bank b, is willing to lend because doing so can potentially increase the repayment of the existing senior loan and help resolve its debt overhang. There is no such benefit for the other bank.[10]

Firms satisfying condition (5) can obtain a loan at the competitive prime rate of zero. Those not meeting the condition obtain a loan at the monopolistic rate at which the bank exploits the firm's entire profit.

To keep the model tractable, we assume that two banks, banks 1 and 2, are operating in the loan market, that the existing senior loan is provided by either bank 1 or bank 2, and that they make lending decisions in two steps:

Step 1: Banks competitively make lending decisions for firms satisfying (5) and which are thus free from debt overhang.

[10]Strictly speaking, a bank without existing exposure to a firm can have incentive to extend a loan to the firm if the bank has exposure to the customers or suppliers of the firm to increase their repayment. To keep the exposition simpler, at this stage, we do not take into account such an indirect debt overhang. This simplification would not substantially affect our theoretical results presented later, although the set of firms subject to the debt overhang will be larger if we explicitly consider the indirect debt overhang.

Step 2: Each bank makes lending decisions noncooperatively for firms not satisfying (5) and which face a debt overhang problem.

We note that a firm might satisfy the eligibility condition for loans in the competitive market in the first step as a result of the lending decisions in the second step. To make the problem tractable, we assume that a firm that becomes able to obtain a loan in the second step cannot return to the first step and try to obtain a loan at the competitive rate.

2.4 Equilibrium

We derive the market equilibrium backwardly by first deriving the equilibrium outcome in the product market and then deriving the optimal refinancing decision of banks.

2.4.1 Profit maximization by each firm

We formulate the firms' price setting as the Nash equilibrium in a noncooperative simultaneous-move game. Each firm sets its price to maximize its profit under the given demand schedules and the prices of other firms and demands from outside the network. The outcome is a system of equations that relate the total sales of firms.

Each firm has a certain level of market power because the products are imperfectly substitutable; thus, it can set the price of its product. The profit-maximization problem for firm i is

$$\max_{p_i} \quad (p_i - p^i)x_i$$

under the demand function for firm i when $e_i = 1$. The demand function for firm i is the sum of the demand from outside the network (4) and the input demand within the network (3); i.e.:

$$x_i = h_i + \sum_{j=1}^{n} e_j x_{ji} = \phi_i p_i^{-\theta} + \sum_{j=1}^{n} \left(\frac{p^j}{p_i}\right)^{\theta} e_j e_i w_{ji} x_j. \tag{6}$$

The first-order condition of the maximization problem provides the best response function of firm i with respect to price:

$$p_i = \frac{\theta}{\theta - 1} p^i, \tag{7}$$

for firm i that is operating; that is, $e_i = 1$.[11] The coefficient $\theta/(\theta - 1)$ is the markup and the profit of firm i is

$$\pi_i = \frac{p_i x_i}{\theta}. \tag{8}$$

[11]We use a convention that if $e_i = 0$, then $p_i = \infty$ yet $p_i e_i = 0$.

2.4.2 Equilibrium outcomes of the product market

The Nash equilibrium of the price-setting game is uniquely determined for any given $\{e_i\}_{i=1}^n$ by solving the system of simultaneous equations of the best response functions of each firm (7). A tractable equation is obtained by raising both sides to the power of $1-\theta$ and plugging the definition of the average cost p^j (1) under the assumption that the price of the input from outside the network is one. The equation describing the equilibrium prices is

$$\mathbf{p}_\theta = \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E}(\mathbf{W}\mathbf{E}\mathbf{p}_\theta + \mathbf{w_0}), \tag{9}$$

where $\mathbf{p}_\theta \equiv (p_1^{-(\theta-1)}, p_2^{-(\theta-1)}, \cdots, p_n^{-(\theta-1)})'$, $\mathbf{w_0} \equiv (w_{10}, w_{20}, \cdots, w_{n0})'$, $\mathbf{W}$ is the $n \times n$ matrix whose (i,j) element is equal to w_{ij}, and $\mathbf{E}$ is a diagonal matrix whose i-th diagonal element is e_i. Note that the system is linear in $p_i^{-(\theta-1)}$. Therefore, a simple matrix calculation gives[12]

$$\mathbf{p}_\theta = \left\{\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{EWE}\right\}^{-1} \mathbf{E}\mathbf{w_0}\left(\frac{\theta-1}{\theta}\right)^{\theta-1}, \tag{10}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. The price level is thus uniquely determined.

Next, we derive sales in equilibrium. By plugging the first-order condition (7) for firm j into p^j in the last term of demand function (6) and multiplying both sides of (6) by p_i, we can express the total sales of firm i as

$$p_i x_i = \phi_i p_i^{1-\theta} + \sum_{j=1}^{n} \left(\frac{p_j}{p_i}\right)^{\theta-1} \left(\frac{\theta-1}{\theta}\right)^{\theta} e_i e_j w_{ji} p_j x_j. \tag{11}$$

Multiplying both sides of (11) by e_i and stacking vertically gives total sales:

$$\mathbf{s} = \mathbf{f} + \mathbf{Q}\mathbf{s}, \tag{12}$$

where $\mathbf{s}$ is a vector of total sales defined as $\equiv (e_1 p_1 x_1, e_2 p_2 x_2, \cdots, e_n p_n x_n)'$; $\mathbf{Q}$ is the $n \times n$ matrix whose (i,j) element is $q_{ij} \equiv \left(\frac{p_j}{p_i}\right)^{\theta-1} \left(\frac{\theta-1}{\theta}\right)^{\theta} e_i e_j w_{ji}$; and $\mathbf{f}$ (sales to outside the network) $\equiv (\phi_1 p_1^{1-\theta}, \phi_2 p_2^{1-\theta}, \cdots, \phi_n p_n^{1-\theta})'$. From input demand function (1), q_{ij} is the ratio of the cost for input good i over total sales of firm j; that is:

$$q_{ij} = \left(\frac{p_j}{p_i}\right)^{\theta-1} \left(\frac{\theta-1}{\theta}\right)^{\theta} e_i e_j w_{ji} = \frac{p_i x_{ji}}{p_j x_j}. \tag{13}$$

By the assumptions on w_{ij} and the definition of p^i, the matrix $\mathbf{I} - \mathbf{Q}$ is invertible.[13] Therefore, the

[12] Note that the matrix inverse in (10) is well-defined because $((\theta-1)/\theta)^{\theta-1} < 1$ and the largest eigenvalue of $\mathbf{W}$ is less than 1 by the assumption that $0 < \sum_{j=1}^n w_{ij} \leq 1$.

[13] Let $\|\cdot\|_1$ denote the norm, defined as $\|\mathbf{Q}\|_1 \equiv \max_j \sum_{i=1}^n \|Q_{ij}\|$. Let λ be an eigenvalue of $\mathbf{Q}$. It is known that

sales vector is uniquely determined and written as

$$\mathbf{s} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{f} = \sum_{k=0}^{\infty} \mathbf{Q}^k \mathbf{f}. \tag{14}$$

The equilibrium in the product market is determined according to price vector (10). The price level determines sales by (14). We note that the inverse matrix $(\mathbf{I} - \mathbf{Q})^{-1}$ is conceptually the same as the Leontief inverse matrix.

We now derive the profits. From the definition of q_{ij}, we obtain the matrix $\mathbf{Q} = \left(\frac{\theta-1}{\theta}\right)^{\theta} \mathbf{P}_\theta \mathbf{E}\mathbf{W}'\hat{\mathbf{P}}_\theta$, where $\mathbf{P}_\theta$ is an $n \times n$ diagonal matrix whose i-th element is $p_i^{-(\theta-1)}$, $\mathbf{E}$ is a diagonal matrix whose i-th element is e_i, and $\hat{\mathbf{P}}_\theta$ is a diagonal matrix whose i-th element is $e_i p_i^{\theta-1}$. Note that $\mathbf{P}_\theta \cdot \hat{\mathbf{P}}_\theta = \mathbf{E}$. Under the abovementioned notations, the Leontief inverse matrix is

$$(\mathbf{I} - \mathbf{Q})^{-1} = \sum_{k=0}^{\infty} \mathbf{Q}^k = \mathbf{P}_\theta \sum_{k=0}^{\infty} \left\{ \left(\frac{\theta-1}{\theta}\right)^{\theta} \mathbf{E}\mathbf{W}' \right\}^k \hat{\mathbf{P}}_\theta = \mathbf{P}_\theta \mathbf{A}^{-1} \hat{\mathbf{P}}_\theta, \tag{15}$$

where

$$\mathbf{A} = \mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta} \mathbf{E}\mathbf{W}'. \tag{16}$$

The vector of sales outside a network, whose i-th element equals $h_i p_i$ (4), is

$$\mathbf{f} = \mathbf{\Phi} \mathbf{P}_\theta \mathbf{e}, \tag{17}$$

where $\mathbf{\Phi}$ is a diagonal matrix whose i-th element is ϕ_i. By plugging (15) and (17) into (14) and then into (8), we obtain the vector of each firm's profit:

$$\pi(\mathbf{e}) = \frac{1}{\theta} \mathbf{P}_\theta \mathbf{A}^{-1} \mathbf{\Phi} \mathbf{e}. \tag{18}$$

2.4.3 Financial market

Last, we discuss the equilibrium of the financial market. Recall that the financial market operates in two steps, where the first step determines the set of firms that obtain loans free from the debt overhang and the second step involves banks deciding whether to provide loans to firms with a debt overhang. In this second step, banks take into account the network effect of a firm's survival, and

$\|\lambda\| \leq \|\mathbf{Q}\|_1$. $\|\mathbf{Q}\|_1 < 1$ holds because:

$$\begin{aligned}\sum_{i=1}^{n} q_{ij} &= \left({p^j}^{-(\theta-1)} - w_{j0} p_0^{-(\theta-1)}\right) \frac{{p^j}^{\theta}}{p_j} e_j \\ &= \frac{p^j}{p_j} \left(1 - \frac{w_{j0} p_0^{-(\theta-1)}}{\sum_{i=0}^{m} e_i w_{ji} p_i^{-(\theta-1)}}\right) e_j.\end{aligned}$$

The first term, the inverse of the markup rate, is smaller than one by (7). The second term is also smaller than one, so it follows that $\|\lambda\| \leq \|\mathbf{Q}\|_1 < 1$.

this may result in network-motivated forbearance lending.

Step 1: Each bank provides a loan to firm i at the competitive prime rate if and only if

$$\pi_i(\mathbf{e}) - l_{ib} \geq f_i, \text{i.e.}, \pi_i(\mathbf{e}) \geq F_i \tag{19}$$

for a given $\mathbf{e}$. Forbearance never emerges in this step since there is no room for a bank to gain from the externality of a borrowing firm. Let us consider the following recursive map, such that

$$\bar{\mathbf{e}} = \mathcal{G}(\mathbf{e}), \tag{20}$$

where the i-th element of $\bar{\mathbf{e}}$ $(n \times 1)$ is given by

$$\bar{e}_i = \mathbb{1}[\pi_i(\mathbf{e}) \geq F_i],$$

where $\mathbb{1}[\cdot]$ is an indicator function that equals one if $[\cdot]$ is true and zero otherwise. Starting from an initial value of $\mathbf{e} = \mathbf{1}$, where $\mathbf{1}$ is a column vector of ones, the banks repeatedly apply this mapping until they obtain a fixed point $\mathbf{e}^\dagger = \mathcal{G}(\mathbf{e}^\dagger)$. We can show that such a fixed point exists by Tarski's fixed-point theorem (Theorem 1 in Tarski, 1955), and there can be multiple equilibria. We denote the set of firms with $e_i^\dagger = 1$ at the fixed point by D.

Proposition 1 *In the first step, there exists a fixed point $\mathbf{e}^\dagger = \mathcal{G}(\mathbf{e}^\dagger)$, that is, a competitive equilibrium, for recursive map (20).*

Proof. See Appendix 2.

Step 2: Each bank makes a lending decision to firms $i \in D^c$, where D^c is the complement set of D. In other words, D^c is the set of firms for which a bank has an existing senior loan and thus becomes a monopolistic lender. This set consists of two subsets: firms with an existing loan from bank 1, denoted by D_1^c, and those with a loan from bank 2, denoted by D_2^c. As the bank can exert monopolistic power on firms with an existing senior loan, the bank imposes a monopolistic interest rate to capture all of the profit of these firms. Thus, from (18), the profit of bank b in this step is:

$$\begin{aligned}\Pi_b(\mathbf{e}) \equiv \Pi_b(\mathbf{e_b}; \mathbf{e_{b'}},) &= \sum_{i \in D_b^c} e_i\{\pi_i(\mathbf{e}) - F_i\} \\ &= \frac{1}{\theta}\mathbf{e_b'}\mathbf{P}_\theta \mathbf{A}^{-1}\mathbf{\Phi}\mathbf{e} - \mathbf{e_b'}\mathbf{F},\end{aligned}$$

where the lending decision by bank b is denoted by $\mathbf{e_b}$, an $n \times 1$ vector whose i-th element is e_i if $i \in D_b^c$ or 0 otherwise. $\mathbf{F}$ is the column vector whose i-th element is F_i.

Each bank decides noncooperatively whether to extend a loan to each firm to maximize its own profit. An additional twist is that it is a noncooperative lending-decision game between two banks

($b = 1, 2$) where lending decisions affect each other's payoff through the externality in the network, which we discuss in detail later. We focus on Nash equilibria in this game.

Bank b's problem to derive the best response is

$$\max_{\mathbf{e_b}} \Pi_b(\mathbf{e_b}; \mathbf{e_{b'}}), \quad (b, b' = 1, 2, \text{ and } b \neq b'). \tag{21}$$

We can prove the existence of pure strategy Nash equilibria and that the set of equilibria is a complete lattice by applying Theorems 1 and 2 in Zhou (1994).

Proposition 2 *The lending decision game in the second step is a supermodular game. The set of Nash equilibria is a nonempty complete lattice.*

Proof. See Appendix 2.

The forbearance condition for bank b is written as follows[14]:

$$\Pi_b(\mathbf{e_b}; \mathbf{e_{b'}}) - F_i \geq \Pi_b(\mathbf{e_{b,-i}}; \mathbf{e_{b'}}) - l_{ib}, \text{ i.e., } \Pi_b(\mathbf{e_b}; \mathbf{e_{b'}}) - \Pi_b(\mathbf{e_{b,-i}}; \mathbf{e_{b'}}) \geq f_i, \tag{22}$$

where $\mathbf{e_{b,-i}}$ is the vector whose j-th ($j \neq i$) element is equal to e_{bj} and i-th element is 0. The left-hand side in the first inequality is the bank's profit from lending to firm i, and the right-hand side is the bank's profit when it declines to lend and instead writes off the existing loan l_{ib}. This condition implies that firm i can obtain a loan in the second step if its marginal contribution to the bank's profit exceeds the cost of refinancing. At a Nash equilibrium, all firms that obtain refinancing must satisfy the forbearance condition (22). This necessary condition for providing a loan is the key condition for the possibility of forbearance lending. We note that, because the set of values that $\{e_i\}_{i=1}^n$ can take is finite, the bank's problem has a solution, although the solution is not guaranteed to be unique.

2.5 Externality of a firm and network-motivated forbearance

We now define forbearance lending in the context of our model and discuss the externality of a firm in the network that induces forbearance lending.

Definition 1 (Forbearance lending) *A bank undertakes forbearance lending if it extends a loan to firm z even if its economic profit is negative, namely:*

$$e_z = 1 \text{ and } \pi_z(\mathbf{e}) - F_z < 0. \tag{23}$$

[14]If a bank is a monopoly in the loan market, the condition is simpler:

$$\Pi(\mathbf{e}) - \Pi(\mathbf{e}_{-i}) \geq F_i,$$

where $\mathbf{e}_{-i}$ is the vector whose elements are the same as those of $\mathbf{e}$ except that e_i is set equal to zero.

Note that F_z is the opportunity cost of refinancing firm z, as we assume the rate of return from the outside opportunity equals zero. We also observe that the rate of return from firm z receiving a forbearance loan is

$$\frac{\pi_z(\mathbf{e})}{F_z} - 1 < 0.$$

Under condition (23), this rate of return is less than zero, the prime rate.

This type of forbearance lending emerges only in the Nash equilibrium in the second step. We define such forbearance lending as *network-motivated forbearance lending.*

Definition 2 (Network-motivated forbearance lending) *Network-motivated forbearance lending from bank b to firm $z \in D_b^c$ is lending that satisfies all of the following conditions: $e_z = 1$,*

$$\begin{aligned} &\textit{and} \quad \pi_z(\mathbf{e}) - F_z < 0, \\ &\textit{and} \quad \Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}}) \geq f_z, \end{aligned} \tag{24}$$

where $\mathbf{e_{-z}}$ is the vector whose i-th element is e_i if $i \neq z$ and the z-th element is 0.

It is important to note that network-motivated forbearance lending emerges when a bank can obtain a sufficient return from other firms in the network to cover the cost of supporting firm z.

Condition (24) suggests that network-motivated forbearance lending can enhance welfare, as it increases bank b's profit by more than the cost of maintaining firm z's operations. Importantly, in this scenario, bank b captures all the profits of firm z, meaning that the value added by firm z is fully reflected in the increase in bank b's profits. Therefore, as long as welfare is defined as the difference between the value added in the supply network and the costs associated with sustaining it, network-motivated forbearance lending contributes to welfare improvement.

Next, to fully understand the externality of a firm throughout a network, we decompose the left-hand side of forbearance condition (24) after plugging in (18), which represents the difference in bank b's profit with or without firm z or, in other words, the contribution of firm $z \in D_b^c$ to the profit of bank b:

$$\begin{aligned} \Pi_b(\mathbf{e_b}) - \Pi_b(\mathbf{e_{b,-z}}) =& \frac{1}{\theta}\mathbf{e_b}'\mathbf{P}_\theta \mathbf{A}^{-1}\mathbf{\Phi e} - \frac{1}{\theta}\mathbf{e_{b,-z}}'\mathbf{P}_{\theta,-\mathbf{z}}\mathbf{A_{-z}}^{-1}\mathbf{\Phi e_{-z}} \\ =& \pi_z(\mathbf{e}) \quad \textbf{(own profit)} && (25) \\ &+ \frac{1}{\theta}\mathbf{e_{b,-z}}'\mathbf{P}_\theta \mathbf{A}^{-1}\mathbf{\Phi}\left(\mathbf{e_b} - \mathbf{e_{b,-z}}\right) \quad \textbf{(demand propagation)} && (26) \\ &+ \frac{1}{\theta}\mathbf{e_{b,-z}}'\mathbf{P}_\theta(\mathbf{A}^{-1} - \mathbf{A_{-z}}^{-1})\mathbf{\Phi e_{-z}} \quad \textbf{(additional link effect)} && (27) \\ &+ \frac{1}{\theta}\mathbf{e_{b,-z}}'\left(\mathbf{P}_\theta - \mathbf{P}_{\theta,-\mathbf{z}}\right)\mathbf{A_{-z}}^{-1}\mathbf{\Phi e_{-z}}, \quad \textbf{(price externality)} && (28) \end{aligned}$$

where $\mathbf{e_{b,-z}}$ is a vector obtained by replacing the z-th element of $\mathbf{e_b}$ with zero, and the subscript "$-z$" of the matrices indicates that they are evaluated at $\mathbf{e_{-z}}$. The first term (25) is the profit of firm z if it receives forbearance lending and continues operating. The other terms represent externalities.

The second term (26) captures the effect of demand propagation to other firms among the borrowers of bank b. This term is nonnegative because $\mathbf{A^{-1}} = \sum_{k=0}^{\infty} \left(\frac{\theta-1}{\theta}\right)^{\theta k} (\mathbf{EW'})^k$ and all elements are nonnegative. It is noteworthy that ϕ_z, the sales of firm z outside the network, matters only in this term. If ϕ_z is zero, that is, firm z sells its product only to other firms in the network, the demand propagation effect (26) is zero, while other effects remain positive. Thus, forbearance lending motivated by demand propagation occurs only for firms that sell their products outside the network or to consumers.

The third term (27) captures the contribution of firm z to other firms through additional links within the network. This term is also always nonnegative in our model because

$$\mathbf{A^{-1}} - \mathbf{A_{-z}^{-1}} = \mathbf{A_{-z}^{-1}}(\mathbf{A_{-z}} - \mathbf{A})\mathbf{A^{-1}} = \mathbf{A_{-z}^{-1}} \left(\frac{\theta-1}{\theta}\right)^{\theta} (\mathbf{E} - \mathbf{E_{-z}})\mathbf{W'A^{-1}} \geq \mathbf{0}.$$

The fourth term (28) is the contribution of firm z to bank b's profit by reducing the product prices of other firms. This is also always nonnegative, as shown in Lemma 1 in Appendix 1. Under our CES production function, a greater variety of inputs improves productivity and reduces the average cost. It also reduces product prices under the constant markup, and this increases sales outside the network as well as the total profit of the network.

The sum of the three terms (26)–(28) is the externality, that is, the contribution to network profit excluding firm z's own profit. The existence of network-motivated forbearance lending is assured by the abovementioned observation that each component of the externality, (26), (27), and (28), is nonnegative.

Proposition 3 (Existence of network-motivated forbearance lending) *All of the components of the externality of firm z, namely, (26), (27), and (28), are nonnegative. For any F_z satisfying $F_z > \pi_z(\mathbf{e})$, there always exists a pair $f_z(> 0)$ and l_z that causes network-motivated forbearance lending; that is:*

$$\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}}) > f_z > \pi_z(\mathbf{e}) - l_z.$$

Proof. Immediate from the fact that each component of the externality, (26), (27), and (28), is nonnegative. □

The above inequality implies that firm z satisfies the forbearance condition for bank b (24) despite the bank's negative return from the loan (23). The wedge between the contribution of firm i to the network profit and firm i's own profit is the externality. The proposition indicates that firms with a larger externality have more opportunities to obtain network-motivated forbearance lending.

2.6 Relation to other types of forbearance lending

The network-motivated forbearance lending described in this paper might appear similar to, but is not to be confused with, other types of forbearance lending described in the literature under assumptions that we do not impose. Importantly, forbearance lending that does not satisfy our forbearance condition (24) is welfare deteriorating, whereas network-motivated forbearance lending is welfare enhancing. In this subsection, we relate our model to these other models of forbearance lending.

2.6.1 Evergreening (e.g., Peek and Rosengren, 2005; Faria-e-Castro et al., 2024)

Evergreening is lending to conceal the loss incurred from a bad loan. Suppose that a bank has an existing loan l_z to firm z. To simplify the analysis, suppose that the bank has no existing loans to other firms in the network. If the bank refuses to lend F_z to the firm, it writes off the loan and suffers a loss of l_z. Additionally, the bank will incur extra costs if this refusal results in a loan loss and a breach of regulatory capital requirements, which may lead to the bank's exclusion from international operations. We assume this additional cost is proportional to the existing debt l_z, or cl_z, where c is a positive constant. If the bank approves an additional loan to the firm, the bank obtains $\pi(\mathbf{e}) - F_z$ from the loan if the bank is the only potential lender due to the firm's debt-overhang problem. Thus, the bank is willing to lend F_z if

$$-(1+c)l_z < \pi(\mathbf{e}) - F_z, \text{ i.e., } f_z - cl_z < \pi(\mathbf{e}).$$

Here, the bank has stronger incentive to provide forbearance lending to salvage its existing loan if the cost c from realizing the loan loss and the size of the existing loan l_z are larger. If we assume that the bank considers the externality, the above condition is replaced with

$$f_z - cl_z < \Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e}_{-\mathbf{z}}).$$

Thus, evergreening can emerge even if it is welfare damaging; that is, $\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e}_{-\mathbf{z}}) < f_z$.

The forbearance mechanism that we identify in this study comes from the externality of a firm in a network among borrowers, whereas evergreening is driven by the cost cl_z of writing off an existing loan and damaging the bank's balance sheet. A clear difference between network-motivated forbearance and evergreening is that network-motivated forbearance lending can emerge for a financially sound bank as well as a financially unsound bank with an insufficient capital adequacy ratio.

2.6.2 Gambling for resurrection or risk shifting (e.g., Bruche and Llobet, 2014)

There has been a discussion that financial institutions near insolvency may resort to hazardous lending practices. To explore such a gambling or risk-shifting motive for forbearance lending, we examine scenarios where a bank's liability includes both deposits (debt) and equity, with deposits protected by deposit insurance. The primary aim of bank managers is to maximize the expected return for shareholders who bear limited liability. We assume here that managers do not have perfect foresight, and so loans to firms are risky assets, where the final demand (ϕ_i) carries a certain level of risk.

Under limited liability and the assumption that the payoff to bank shareholders is zero if the bank becomes insolvent, the shareholders of bank b approve a loan to firm $z \in D_1^c$ if

$$Prob(\text{bank } b \text{ is solvent})(E[\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}})|\text{bank } b \text{ is solvent}] - f_z) > 0;$$

that is,

$$E[\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}})|\text{bank } b \text{ is solvent}] > f_z.$$

In other words, the bank cares only about the upside of the equity value and ignores the downside because the loss is covered by deposit insurance.

Lending under this condition is value-destroying if

$$E[\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}})| \text{ bank } b \text{ is solvent}] > f_z > E[\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}})].$$

If the bank has sufficient equity and is always solvent, the gap disappears because $E[\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}})| \text{ bank is solvent}] = E[\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}})]$. Thus, this gap emerges only for a financially unsound bank that maintains insufficient capital and can become insolvent. In other words, only financially unhealthy banks tend to engage in such risky and value-destroying forbearance lending.

Risk shifting may appear in the form of covariance risk shifting (Landier et al., 2015), i.e., risk taking by holding more assets whose returns are highly positively correlated with the existing assets. Since the return from the loans to suppliers or customers of a borrower is likely to be correlated with the return from the loan to the borrower, the network-motivated forbearance and the covariance risk shifting generate an observationally equivalent lending behavior. It is an empirical challenge to discern between lending behaviors in actual data. A clear difference between our theory and the risk-shifting hypothesis is that a bank has an incentive to provide a network-motivated forbearance irrespective of the bank's financial soundness. This prediction provides a clue to overcoming this empirical challenge.

2.7 Identifying welfare-damaging forbearance

While network-motivated forbearance lending improves welfare and evergreening or gambling is welfare damaging, in practice, these mechanisms can all be active simultaneously. In the main part of our analysis, we focus on network-motivated forbearance lending; however, because the welfare (and thus policy) implications of these various types of forbearance lending are different, policymakers must determine which types of forbearance lending are most likely to occur so that they can understand the implications and develop appropriate policies. In this subsection, we identify situations in which welfare-destructive forbearance lending is likely to occur.

To identify inefficient lending, the assumptions for these welfare-damaging forbearances suggest that it is helpful to consider bank characteristics. For instance, forbearance lending from an undercapitalized bank is more likely to be damaging to welfare (e.g., Peek and Rosengren, 2005). This observation is in sharp contrast to network-motivated forbearance, which can emerge without these bank-specific factors. The coexistence of this factor with the network motivation also suggests the possibility that evergreening or gambling for resurrection may unintentionally improve the network's profit because of the externality.

In particular, policymakers should consider the financial condition of banks and the influence of firms in the supply network to evaluate the economic cost of forbearance lending. We propose two influence measures in Section 3.

2.8 Alternative financial market structures

Network-motivated forbearance lending emerges if a bank has sufficient market power to obtain returns from loans to peripheral firms to cover the cost of forbearance lending to a core firm. Without this market power, the lending competition becomes a Bertrand competition where the interest rate is set at the prime rate and bank profits are zero, resulting in no bank providing network-motivated forbearance lending in this situation. Thus far, we have focused on firms for whom only the bank with a large existing exposure is willing to provide a loan due to the debt-overhang problem. In this subsection, we provide other realistic examples whereby a bank has market power over peripheral firms.[15]

2.8.1 Relational lending

Another realistic setting, in which a bank has market power over peripheral firms, is relational lending. It is widely recognized that a bank can earn an excess return from lending relationships, which can generate an information advantage over rival banks (Sharpe, 1990; Rajan, 1992) or the ability to provide relation-specific value-adding services using this information advantage (Boot and Thakor, 2000). Many empirical studies provide evidence of this in the financing of small businesses (Degryse and van Cayseele, 2000; Ioannidou and Ongena, 2010), which are presumably peripheral in the supply network described here. These results imply that relationship banking enhances the ability of a bank to recoup the cost of a forbearance loan from peripheral borrowers. Thus, banks that maintain lending relationships with a sufficiently large part of a supply network around an influential firm have greater incentive to engage in network-motivated forbearance lending.

2.8.2 A coalition among oligopolistic banks

Another possibility is a coalition comprised of all oligopolistic banks. A single deviant from the coalition can break this coalition by poaching loans to peripheral firms that benefit from the network

[15] In addition to the examples described here, theoretically possible arrangements that have the same effect as forbearance lending include M&A among firms within a network.

externality and can thus provide the bank a sufficient return to cover the cost of supporting a distressed influential firm. In this case, bank profits are zero, whereas bank profits from network-motivated forbearance facilitated by the coalition are positive. Thus, there is room to form such a coalition.

This coordination problem provides an example of the collective investment decision problem faced by investors connected through a financial network, which is analyzed by Leitner (2005). A bank providing loans to fewer firms in the network around the target of forbearance is less willing to engage in forbearance lending because it expects less benefit from the externality of the forbearance. Despite this difficult situation, network-motivated forbearance lending may be possible by transferring benefits of the externality from banks that gain more to those that gain less, as shown by Leitner (2005), if the forbearance condition for the monopolistic bank (24) is satisfied. A practical form of this benefit transfer, for example, is a loan from banks expecting more benefits to banks expecting fewer benefits at a rate lower than the prime rate.

3 Measures related to network-motivated forbearance lending and empirical predictions

In this section, we propose two measures of the extent of the externality of each firm. We illustrate through stylized examples how these measures and components of externality are related and also propose several empirical hypotheses from our model.

3.1 Indicators of externality

Indicators of the extent of the externality of each firm are convenient for identifying a firm with a large externality in an actual network. We propose two such indicators: the *demand influence coefficient* and the *price (supply) influence coefficient.* The first works as an indicator of the demand propagation of a firm, and the second works as an indicator of the price externality of a firm.

3.1.1 Demand influence coefficient

The *demand influence coefficient* measures the influence of a firm as a buyer among multiple bank clients within a network. The vector of the demand influence coefficient $\mathbf{v_b}$ of each borrower of bank b is defined by

$$\mathbf{v}'_\mathbf{b} \quad \equiv \quad \mathbf{e}'_\mathbf{b}(\mathbf{I}-\mathbf{Q})^{-\mathbf{1}} = \mathbf{e_b}'\sum_{k=0}^{\infty}\mathbf{Q}^k = \mathbf{e_b}'\mathbf{P}_\theta\mathbf{A}^{-\mathbf{1}}\hat{\mathbf{P}}_\theta, \tag{29}$$

where $\mathbf{e_b}$ is a vector whose i-th element is e_i if firm i is a borrower of bank b, or zero otherwise. The last expression comes from (15). The i-th element of $\mathbf{v_b}$, denoted by v_i, measures the centrality of firm i as a buyer. Each element of this vector captures the bank-level aggregation of the demand influence of a firm. Note that our demand influence coefficient is different from the consumer centrality proposed by Baqaee (2018), in that Baqaee's centralities capture the extent of the vulnerability of each firm to a demand shock in the final goods sector; however, our demand centrality captures the extent of the contribution of a firm to the demand for other firms' products.

Firm i's demand influence coefficient represents the influence of firm i on network profit. This observation comes from the fact that, by equations (8) and (14), the total profit of the network is θ^{-1} times $\mathbf{e_b's} = \mathbf{e_b'(I - Q)^{-1}f} = \mathbf{v_b'f}$. To express this in words, the demand influence coefficient v_i measures the influence of firm i's sales outside the network on the total network sales; that is:

$$v_i = \frac{\Delta \text{total sales of the network of bank } b \text{ borrowers}}{\Delta \text{sales of firm } i \text{ outside the network}}.$$

Importantly, the demand influence coefficient considers not only the first-order impact (i.e., the impact on the adjacent neighbors) but also all the higher-order impacts. In the summation in (29), $\mathbf{Q}$ captures the first-order impact, $\mathbf{Q}^2$ captures the second-order impact, and so forth.

The next proposition relates the demand influence coefficient to demand propagation (see Online Appendix 1 for the proof).

Proposition 4 *Demand propagation component* (26) *is*

$$\left(v_z - A_{zz}^{(-1)}\right) \frac{1}{\theta} p_z^{-(\theta-1)} \phi_z,$$

where $A_{zz}^{(-1)}$ *is the* (z, z)*-th element of* $\mathbf{A}^{-1}$*.* $A_{zz}^{(-1)} = 1$ *holds when the* z*-th column of* $\mathbf{W}$ *is zero or* $\mathbf{A}$ *is a triangular matrix. In those cases, the demand propagation component is proportional to* $v_z - 1$*.*

This brings us to the following empirical prediction.

Empirical Prediction 1 *Network-motivated forbearance lending is more often given to a firm that has a higher demand influence coefficient and sells large quantities of its products outside its network or to consumers.*

The definition of the demand influence coefficient $\mathbf{v_b}$ in (29) indicates that an estimate of $\mathbf{v_b}$ is obtained once an estimate of $\mathbf{Q}$ is available. Its (i, j) element, q_{ij}, equals the ratio of the cost of the input good i over the total sale of firm j (see (13)). Thus, we can calculate $\mathbf{Q}$ directly if information regarding the amount of each transaction among firms and the sales of each firm is available. If information about the amount of each transaction among firms is not available but information on total sales and the list of sellers and buyers are both available, we may estimate $\mathbf{Q}$ by applying a spatial autoregressive model to (12), which shows the linear relationship between the sales of each firm and the sales of directly connected firms.

3.1.2 Price (supply) influence coefficient

The price (supply) influence coefficient captures the influence of a firm as an input supplier in a network. Firms affect the product prices of other firms through the supply chain network. To see this, we premultiply both sides of the vector of $p_i^{-(\theta-1)}$ in (10) by $\mathbf{e}_{\mathbf{b}}'$; that is, we calculate a price index of the products of borrowers of bank b:

$$\mathbf{e}_{\mathbf{b}}'\mathbf{p}_\theta = \mathbf{e}_{\mathbf{b}}'\left\{\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right\}^{-1}\mathbf{Ew_0}\left(\frac{\theta-1}{\theta}\right)^{\theta-1} = \boldsymbol{\xi}_{\boldsymbol{b}}'\mathbf{Ew_0}\left(\frac{\theta-1}{\theta}\right)^{\theta-1},$$

where

$$\boldsymbol{\xi}_{\boldsymbol{b}}' \equiv \mathbf{e}_{\mathbf{b}}'\left\{\mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}\right\}^{-1}. \tag{30}$$

The higher the i-th element of the vector $\boldsymbol{\xi}_{\boldsymbol{b}}$, ξ_i, the higher the contribution of firm i to the outside sales of other borrowers in the network due to lower production costs and product prices. We thus call ξ_i firm i's price influence coefficient. The closure of a firm with a high ξ_i is more damaging to borrowers and banks.

This reasoning brings us to another empirical hypothesis.

Empirical Prediction 2 *Network-motivated forbearance more often emerges for a firm with a higher price influence coefficient.*

This influence comes from a property of the CES production function, that a greater variety of inputs generates greater efficiency. Note that this coefficient is different from the supply influence coefficient of Baqaee (2018), in that Baqaee (2018) measures the susceptibility of a firm to a productivity shock in the final goods sector, whereas our coefficient captures the contribution of a firm to network profit by reducing product prices.

The definition (30) indicates that we need the estimate of $\left(\frac{\theta-1}{\theta}\right)^{\theta-1}\mathbf{EWE}$ for the empirical measurement of $\boldsymbol{\xi}_{\boldsymbol{b}}$. If the price information of each firm and the list of sellers and buyers for each firm are available, we can estimate this by applying a spatial autoregressive model to (9).

3.2 Typical network examples: Large buyer

In this subsection, we present an example to illustrate the decomposition of the contribution of firm z to the network profit as well as the relationship between the terms in the decomposition and the demand and supply influence coefficients. We consider a "large buyer" case, where only one firm buys inputs from other firms, and other firms use only good 0 as their input. To keep the exposition simple, we assume a monopolistic bank in this example. The details of the derivations are described in Online Appendix 3.

In this example, firm 1 uses all goods produced in the network as inputs, but other firms do not use any intermediate goods so that firm 1 is the large buyer. Specifically, we set $\mathbf{W} = a\mathbf{u}_1\mathbf{1}_{-1}'$, where a is a scalar satisfying $0 < a < 1/n$, $\mathbf{u}_1$ is the first unit vector, and $\mathbf{1}_{-1}$ is a vector with the first element being equal to 0 and other elements being equal to 1.

The decomposition of the contribution of firm 1, the large buyer, is

$$\begin{aligned}\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-1}}) =& \frac{1}{\theta}\phi_1 \left(w_{10} + a \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{w}_0'\mathbf{e}_{-1} \right) \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \qquad \textbf{(own profit)} \\ &+ \frac{1}{\theta} a \left(\frac{\theta-1}{\theta}\right)^{2\theta-1} \phi_1 \mathbf{w}_0'\mathbf{e}_{-1} \quad \textbf{(demand propagation)} \\ &+ 0 \quad \textbf{(additional link effect)} \quad + 0 \quad \textbf{(price externality)}.\end{aligned}$$

We see that the externality of firm 1 is completely determined by the demand propagation effect, which depends on ϕ_1, so the larger the demand for good 1 from the outside, the stronger the demand propagation effect. We also observe that there is no additional link effect or price externality effect, so the prices of goods $i \geq 2$ are not affected by the presence of firm 1.

These observations are consistent with the implications from the influence coefficients. The demand influence coefficient of firm 1 is

$$v_1 = 1 + a \left(\frac{\theta-1}{\theta}\right)^{2\theta-1} \mathbf{w}_0'\mathbf{e}_{-1} p_1^{\theta-1} > 1,$$

and $v_1 - 1$ is proportional to the demand propagation effect. The price influence coefficient is $\xi_1 = 1$, which is consistent with no price externality effect.

Next, we consider the decomposition of the contribution of firm $z \geq 2$:

$$\begin{aligned}\Pi_b(\mathbf{e}) - \Pi_b(\mathbf{e_{-z}}) =& \frac{1}{\theta} w_{z0} \left(\phi_z + a e_1 \phi_1 \left(\frac{\theta-1}{\theta}\right)^{\theta} \right) \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \qquad \textbf{(own profit)} \\ &+ 0 \quad \textbf{(demand propagation)} \quad + 0 \quad \textbf{(additional link effect)} \\ &+ \frac{1}{\theta} \left(\frac{\theta-1}{\theta}\right)^{2\theta-2} a \phi_1 e_1 w_{z0} \quad \textbf{(price externality)}.\end{aligned}$$

The externality of firm z is completely determined by the price externality effect. Good z is used as an intermediate input for firm 1, and the existence of firm z lowers the price of Good 1, which in turn increases the demand from the outside and increases network profits. There is no demand propagation effect or additional link effect.

The demand influence coefficient is $v_z = 1$, which is consistent with the absence of the demand propagation effect. The supply influence coefficient is:

$$\xi_z = 1 + a e_1 \left(\frac{\theta-1}{\theta}\right)^{\theta-1},$$

which is larger than 1 as long as $e_1 = 1$, and $\xi_z - 1$ is proportional to the price externality effect.

3.3 Determinants of externality

To elucidate the characteristics of firms that are more prone to being a target of network-motivated forbearance, in this subsection, we conduct the comparative statics of the externality (26)–(28) with respect to exogenous parameters.

We focus on the row sums and the column sums of $\mathbf{W}$ and the elasticity of substitution θ. Some comparative statics are difficult to show analytically due to the feature of our model that each element of $\mathbf{W}$ determines both the demand influence coefficients and the price influence coefficients of all firms in the network. We resort to simulations for this difficult part of the comparative statics, which are presented in Online Appendix A4.2.

3.3.1 Input dependence within a network: z-th row of W

The z-th row of $\mathbf{W}$ indicates the input dependence of firm z. The next proposition states that the z-th row of $\mathbf{W}$ is closely related to the demand propagation effect of firm z and that the demand influence coefficient of firm z is increasing in the z-th row elements of $\mathbf{W}$ under some conditions (see Online Appendix 1 for the proof).

Proposition 5 (z-th row of W, v_z, and demand propagation) *1. If all elements in the z-th row of $\mathbf{W}$ are zero, the demand influence coefficient of firm z, v_z, equals one, and the demand propagation effect* (26) *and the additional link effect* (27) *are both zero.*

2. Suppose that the matrix of the weights of the production functions has the form $\mathbf{W} + a\mathbf{u}_z\mathbf{w}'_{z\cdot}$, where all elements in the z-th column of $\mathbf{W}$ are zero, a is a constant in $[0, 1]$, $\mathbf{w}_{z\cdot}$ is a vector of positive numbers except the z-th element, which equals zero, and $\mathbf{u}_z$ is the z-th unit vector. The demand influence coefficient, v_z, is increasing in a.

The first part of the proposition indicates that the input dependence of firm z on other firms in the network is the necessary condition for the existence of the demand propagation effect (26) and the additional link effect (27) and that the demand influence coefficient has a value greater than one only when the firm purchases inputs from a firm in the network.

In the second part, we note that the z-th column elements of the matrix $\mathbf{W} + a\mathbf{u}_z\mathbf{w}'_{z\cdot}$ are all zero and that its z-th row elements are increasing in a. The demand influence coefficient increases as firm z uses more inputs from other firms, provided that no other firm uses good z as its input.

3.3.2 Versatility within a network: z-th column of W

In contrast to the z-th row, the z-th column of $\mathbf{W}$ is related to the externality through price (28). The following proposition states that the z-th column of $\mathbf{W}$ is closely related to the price externality of firm z and that the price influence coefficient of firm z is increasing in the z-th column elements of $\mathbf{W}$ under certain conditions (see Online Appendix 1 for the proof).

Proposition 6 (z-th column of W, ξ_z, and price externality) *1. If all elements in the z-th column of* $\mathbf{W}$ *are zero, the price (supply) influence coefficient of firm z,* ξ_z*, equals one, and the price externality* (28) *is zero.*

2. Suppose that the matrix of the weights in the production function has the form $\mathbf{W} + a\mathbf{w}_{\cdot z}\mathbf{u}'_z$*, where all elements in the z-th row of* $\mathbf{W}$ *are zero; a is a constant in* $[0, 1]$*;* $\mathbf{w}_{\cdot z}$ *is a vector of positive numbers, except the z-th element, which equals zero; and* $\mathbf{u}_z$ *is the z-th unit vector. The supply influence coefficient of firm z is increasing in a.*

The first part of this proposition indicates that the input dependence of other firms in the network on the product of firm z is the necessary condition for the existence of price externality (28). The price influence coefficient has a value greater than one only when the firm's product is used by other firms in the network. Otherwise, the price influence coefficient is one, which is the lowest value.

In the second part, we note that the z-th row of the matrix $\mathbf{W} + a\mathbf{w}_{\cdot z}\mathbf{u}'_z$ is a zero vector and that the z-th column elements are increasing in a. The supply influence coefficient increases as more firms use good z as their input, provided that firm z does not use the products of other firms as its input.

3.3.3 Substitutability (competition): θ

The elasticity of substitution θ determines the extent of competition in the product market. If θ is closer to one (i.e., the products are less substitutable), the markup $\theta/(\theta - 1)$ becomes larger. In this situation, it is easy for a firm to internalize all positive externalities for its customers as its own profit. If, instead, θ is very large so that the products are highly substitutable, then the profit becomes closer to zero. We summarize the effect of product substitutability in the following proposition (see Online Appendix 1 for the proof).

Proposition 7 (Elasticity of substitution θ and externality) *1. As θ gets closer to one and products are highly differentiated, the demand propagation effect* (26) *and the additional link effect* (27) *disappear, while the price externality* (28) *remains positive.*

2. As θ increases to ∞, all components of externality (26)–(28) *converge to zero.*

This result leads to the following empirical prediction.

Empirical Prediction 3 *Network-motivated forbearance is more often observed in a sector with a moderate level of product differentiation and markup.*

3.4 Further empirical predictions

In addition to the hypotheses proposed so far, several additional testable hypotheses can be derived. First, from the basic mechanism of network-motivated forbearance lending, the cost to support an influential firm is recouped from the returns from loans to peripheral firms. This is expressed in the following empirical prediction:

Empirical Prediction 4 *Network-motivated forbearance lending for an influential firm with a rate of return lower than the prime rate is observed jointly with a higher rate of return on lending to peripheral firms.*

Second, since the externality perceived by a bank, expressed as components (26), (27), and (28), is increasing in $\mathbf{e_b}$ (so that the benefit from the externality is larger for a bank that covers a larger part of a network), this leads to the following empirical prediction.

Empirical Prediction 5 *Network-motivated forbearance lending is found more often for a large or dominant bank.*

Third, as shown in the Online Appendix A4.3, the externality is larger for a more densely connected network, so firms opt to use the same bank to benefit from network-motivated forbearance lending. This leads to the following hypothesis:

Empirical Prediction 6 *Firms in a more densely connected supply network tend to borrow from the same bank to benefit from network-motivated forbearance in the future.*

Lastly, the argument on evergreening and gambling for resurrection described in Section 2.6 brings us the following hypothesis:

Empirical Prediction 7

Forbearance lending provided by an undercapitalized bank is more likely to be welfare damaging.

4 Concluding remarks

In this paper, we have formulated a theoretical model of the optimal refinancing decision of a banking sector facing a supply chain network of firms. From this model, we have identified the components of the positive externality of a firm to the supply network (demand propagation, price externality, and additional link effect) and have constructed two indicators of the extent of these externalities: the demand influence coefficient and the price influence coefficient. We have also shown that a profit-maximizing bank is motivated to provide forbearance lending to an influential buyer or an influential seller. Our analysis sheds light on the existence of welfare-improving forbearance lending by financially sound banks to influential but loss-making firms. Our analysis provides insight on several policy questions. For example, the suspension of economic activities and supply chain disruptions due to the COVID-19 pandemic in 2020–2021 spurred governments to provide rescue loans to those firms hit by this shock of unprecedented magnitude. The estimated demand and price influence coefficients derived in this study provide a guideline for selecting the targets of such rescue lending.

In order to describe network-motivated forbearance lending in a parsimonious manner, we have made several simplifying assumptions. First, we have assumed that the network is fixed, which is valid for a short-run analysis or in an industry that requires the accumulation of relation-specific investment and information to improve productivity. This would apply, for example, to the automotive and financial sectors, where potential entrants are not likely to be more efficient than incumbent firms. However, it may not generally be realistic for a long-run analysis, where firms can switch relationships more frequently. In this context, network-motivated forbearance lending may be efficient in the short run but might hinder the entry of more efficient and productive new firms or impede the adaptation of new technologies by firms supported by forbearance, making it detrimental in the long run. Second, we have assumed that the demand for funds by each firm is fixed. Generally, however, the higher interest rate imposed on peripheral firms to cover the cost of forbearance for a hub firm can reduce their production, which may in turn alter their need for funding. Third, we have not included the possibility of moral hazard of an influential firm expecting forbearance lending. Identifying the relative importance of these countervailing factors and their effect on the extent of welfare-improving network-motivated forbearance lending remains an important empirical question for the future.

Appendix 1: Lemmas to prove propositions

Let $\mathbf{E_1}$ be the $n \times n$ diagonal matrix whose i-th diagonal element is e_i for $i \neq z$, and $e_z = 1$. Let $\mathbf{E_0}$ be the $n \times n$ diagonal matrix whose i-th diagonal element is e_i for $i \neq z$, and $e_z = 0$. The subscript "1" of a matrix indicates that the matrix is evaluated at $\mathbf{E_1}$, and the subscript "0" indicates it is evaluated at $\mathbf{E_0}$.

First, we show that the price of each product is nonincreasing in the number of operating firms. This holds because of the love-of-variety assumption in our production function. A higher variety of inputs improves productivity and reduces product prices.

Lemma 1 $p_i^{-(\theta-1)}$ *($i = 1, 2, \cdots, n,\ \theta > 1$) is nondecreasing in the number of operating firms; namely, each product price is nonincreasing in the number of operating firms.*

Proof. From (10), we have

$$\begin{aligned}\mathbf{p}'_{\theta\mathbf{1}} - \mathbf{p}'_{\theta\mathbf{0}} &\propto \mathbf{w}'_{\mathbf{0}}(\mathbf{E_1}\mathbf{B_1^{-1}} - \mathbf{E_0}\mathbf{B_0^{-1}}) \\ &= \mathbf{w}'_{\mathbf{0}}\{\mathbf{E_1}\mathbf{B_1^{-1}}(\mathbf{B_0} - \mathbf{B_1})\mathbf{B_0^{-1}} + (\mathbf{E_1} - \mathbf{E_0})\mathbf{B_0^{-1}}\} \geq \mathbf{0},\end{aligned}$$

where

$$\mathbf{B}_{e_z} \equiv \left\{ \mathbf{I} - \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{E_{e_z}}\mathbf{W}\mathbf{E_{e_z}} \right\},$$

and "$\geq \mathbf{0}$" indicates that every element of a matrix is nonnegative. This holds because all elements in $\mathbf{B_{e_z}}, \mathbf{B_{e_z}^{-1}}, \mathbf{E_1} - \mathbf{E_0}$, and $\mathbf{w_0}$ are nonnegative by definition. As

$$\begin{aligned}\mathbf{B_0} - \mathbf{B_1} &\propto \mathbf{E_1 W' E_1} - \mathbf{E_0 W' E_0} \\ &= (\mathbf{E_1} - \mathbf{E_0})\mathbf{W'E_1} + \mathbf{E_0 W'}(\mathbf{E_1} - \mathbf{E_0}) \geq \mathbf{0},\end{aligned} \tag{31}$$

every element of the first term in the bracket is also nonnegative. Thus,

$$\mathbf{p}'_{\theta 1} \geq \mathbf{p}'_{\theta 0}. \qquad \square$$

Lemma 2 *The profit before repayment of an operating firm i, $\pi_i(\mathbf{e})$ ($i = 1, 2, \cdots, n$), is nondecreasing in the number of operating firms in the network.*

Proof. From (18), the column vector of firm profit $\pi(\mathbf{e})$ whose i-th element is $\pi_i(\mathbf{e})$ ($i = 1, 2, \cdots, n$) is given by:

$$\pi(\mathbf{e}) = \frac{1}{\theta}\mathbf{P}_\theta \mathbf{A^{-1}\Phi e}.$$

Let us denote $\mathbf{E_0 1}$ and $\mathbf{E_1 1}$ by $\mathbf{e_0}$ and $\mathbf{e_1}$, respectively. It is sufficient to show that every element of the following vector is nonnegative.

$$\begin{aligned}\pi(\mathbf{e_1}) - \pi(\mathbf{e_0}) =& \frac{1}{\theta}\mathbf{P}_{\theta 1}\mathbf{A_1^{-1}\Phi e_1} - \frac{1}{\theta}\mathbf{P}_{\theta 0}\mathbf{A_0^{-1}\Phi e_0} \\ =& \frac{1}{\theta}\left\{(\mathbf{P}_{\theta 1} - \mathbf{P}_{\theta 0})\mathbf{A_0^{-1}\Phi e_0} + \mathbf{P}_{\theta 1}\mathbf{A_1^{-1}}(\mathbf{A_0} - \mathbf{A_1})\mathbf{A_0^{-1}\Phi e_0} + \mathbf{P}_{\theta 1}\mathbf{A_1^{-1}\Phi}(\mathbf{e_1} - \mathbf{e_0})\right\}.\end{aligned}$$

All the elements in $\mathbf{A_0^{-1}}$ and $\mathbf{A_1^{-1}}$ are nonnegative since $\mathbf{A_0^{-1}} = \sum_{k=0}^{\infty}\left\{\left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E_0 W'}\right\}^k$ and $\mathbf{A_0^{-1}} = \sum_{k=0}^{\infty}\left\{\left(\frac{\theta-1}{\theta}\right)^\theta \mathbf{E_1 W'}\right\}^k$, the summation of matrices of nonnegative elements.

Thus, the last expression is nonnegative by Lemma 1, and

$$\mathbf{A_0} - \mathbf{A_1} = \left(\frac{\theta - 1}{\theta}\right)^\theta (\mathbf{E_1} - \mathbf{E_0})\mathbf{W'} \geq \mathbf{0}. \tag{32}$$

Thus, the difference is nonnegative. $\square$

Lemma 3 *The bank profit before subtracting the costs of financing, $\Pi_b(\mathbf{e})$, is increasing in the number of operating firms in the network.*

Proof. By Lemma 2, $\Pi_b(\mathbf{e}) = \sum_{i \in D_b^c} e_i \pi_i(\mathbf{e})$ is increasing in the number of operating firms. $\square$

Lemma 4 *The set of $\mathbf{e}$ equipped with a partial order, where the order for each pair of $\mathbf{e}$ and $\hat{\mathbf{e}} \in \{0,1\}^n$ is defined by $\mathbf{e} \geq \hat{\mathbf{e}}$ if $e_i \geq \hat{e}_i$ for any $i = 1, 2, \cdots, n$, is a complete lattice.*

Proof. Let us define the lattice operations for $\mathbf{e}, \hat{\mathbf{e}} \in \{0,1\}^n$,

$$\mathbf{e} \vee \hat{\mathbf{e}} \equiv (\max[e_1, \hat{e}_1], \max[e_2, \hat{e}_2], \cdots, \max[e_n, \hat{e}_n]),$$
$$\mathbf{e} \wedge \hat{\mathbf{e}} \equiv (\min[e_1, \hat{e}_1], \min[e_2, \hat{e}_2], \cdots, \min[e_n, \hat{e}_n]).$$

There exist $\mathbf{e} \vee \hat{\mathbf{e}}$ and $\mathbf{e} \wedge \hat{\mathbf{e}} \in \{0,1\}^n$ for any pair of $\mathbf{e}$ and $\hat{\mathbf{e}} \in \{0,1\}^n$. Thus, the set of $\mathbf{e}$ with the partial order is a lattice. Moreover, it is a complete lattice because the supremum $\vee\mathcal{S}$ and the infimum $\wedge\mathcal{S}$ exist for any subset $\mathcal{S} \subseteq \{0,1\}^n$. $\square$

Lemma 5 *The firm profit difference $\pi(\mathbf{e_1}) - \pi(\mathbf{e_0})$ and the bank profit difference $\Pi_b(\mathbf{e_1}) - \Pi_b(\mathbf{e_0})$ resulting from switching e_i $(i = 1, 2, \cdots, n)$ from 0 to 1 are nondecreasing in e_j $(j \neq i)$.*

Proof. Take firm y $(y \neq z)$. Let $\mathbf{E_{11}}$ and $\mathbf{E_{01}}$ be $\mathbf{E_1}$ and $\mathbf{E_0}$ with $e_y = 1$, respectively. Likewise, let $\mathbf{E_{10}}$ and $\mathbf{E_{00}}$ be $\mathbf{E_1}$ and $\mathbf{E_0}$ with $e_y = 0$, respectively. We denote $\mathbf{e_{11}} = \mathbf{E_{11}1}, \mathbf{e_{01}} = \mathbf{E_{01}1}, \mathbf{e_{10}} = \mathbf{E_{10}1}$, and $\mathbf{e_{00}} = \mathbf{E_{00}1}$. We denote the price matrix $\mathbf{P}_\theta$ corresponding to each of the four cases by the subscript 11 (firms z and y operate), 01 (firm z does not operate but firm y operates), 10 (firm z operates but firm y does not), and 00 (both do not operate), respectively. It is sufficient to show that the following difference is nonnegative. $\pi(\mathbf{e})$ is the vector of the profit of each firm, whose $i-$th element is $\pi_i(\mathbf{e})$.

$$\begin{aligned}
&\pi(\mathbf{e_{11}}) - \pi(\mathbf{e_{01}}) - \{\pi(\mathbf{e_{10}}) - \pi(\mathbf{e_{00}})\} && (33)\\
&= \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{11}}\mathbf{A_{11}^{-1}}\mathbf{\Phi}(\mathbf{e_{11}} - \mathbf{e_{01}}) - \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{10}}\mathbf{A_{10}^{-1}}\mathbf{\Phi}(\mathbf{e_{10}} - \mathbf{e_{00}}), && (34)\\
&+ \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{11}}(\mathbf{A_{11}^{-1}} - \mathbf{A_{01}^{-1}})\mathbf{\Phi}\mathbf{e_{01}} - \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{10}}(\mathbf{A_{10}^{-1}} - \mathbf{A_{00}^{-1}})\mathbf{\Phi}\mathbf{e_{00}}, && (35)\\
&+ \frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{11}} - \mathbf{P}_{\theta\mathbf{01}})\mathbf{A_{01}^{-1}}\mathbf{\Phi}\mathbf{e_{01}} - \frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{10}} - \mathbf{P}_{\theta\mathbf{00}})\mathbf{A_{00}^{-1}}\mathbf{\Phi}\mathbf{e_{00}}. && (36)
\end{aligned}$$

The term (34) can be expanded as follows:

$$\begin{aligned}
&\frac{1}{\theta}\mathbf{P}_{\theta\mathbf{11}}\mathbf{A_{11}^{-1}}\mathbf{\Phi}(\mathbf{e_{11}} - \mathbf{e_{01}}) - \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{10}}\mathbf{A_{10}^{-1}}\mathbf{\Phi}(\mathbf{e_{10}} - \mathbf{e_{00}})\\
&= \frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{11}} - \mathbf{P}_{\theta\mathbf{10}})\mathbf{A_{11}^{-1}}\mathbf{\Phi}(\mathbf{e_{11}} - \mathbf{e_{01}}) + \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{10}}\mathbf{A_{11}^{-1}}(\mathbf{A_{10}} - \mathbf{A_{11}})\mathbf{A_{10}^{-1}}\mathbf{\Phi}(\mathbf{e_{11}} - \mathbf{e_{01}})\\
&+ \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{10}}\mathbf{A_{10}^{-1}}\mathbf{\Phi}(\mathbf{e_{11}} - \mathbf{e_{01}} - \mathbf{e_{10}} + \mathbf{e_{00}}).
\end{aligned}$$

The first term is nonnegative by Lemma 1. The second term is also nonnegative by (32). The third term is zero. Thus, term (34) is nonnegative.

Term (35) can be expanded as follows.

$$
\begin{aligned}
&\frac{1}{\theta}\mathbf{P}_{\theta\mathbf{11}}(\mathbf{A}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{01}}^{-\mathbf{1}})\boldsymbol{\Phi}\mathbf{e}_{\mathbf{01}} - \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{10}}(\mathbf{A}_{\mathbf{10}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{00}}^{-\mathbf{1}})\boldsymbol{\Phi}\mathbf{e}_{\mathbf{00}} \\
&= \frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{11}} - \mathbf{P}_{\theta\mathbf{10}})(\mathbf{A}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{01}}^{-\mathbf{1}})\boldsymbol{\Phi}\mathbf{e}_{\mathbf{01}} + \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{10}}(\mathbf{A}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{01}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{10}}^{-\mathbf{1}} + \mathbf{A}_{\mathbf{00}}^{-\mathbf{1}})\boldsymbol{\Phi}\mathbf{e}_{\mathbf{01}} \\
&+ \frac{1}{\theta}\mathbf{P}_{\theta\mathbf{10}}(\mathbf{A}_{\mathbf{10}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{00}}^{-\mathbf{1}})\boldsymbol{\Phi}(\mathbf{e}_{\mathbf{01}} - \mathbf{e}_{\mathbf{00}}).
\end{aligned}
$$

Each element of $\mathbf{A}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{01}}^{-\mathbf{1}}$ is nonnegative by (32) because $\mathbf{A}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{01}}^{-\mathbf{1}} = \mathbf{A}_{\mathbf{11}}^{-\mathbf{1}}(\mathbf{A}_{\mathbf{01}} - \mathbf{A}_{\mathbf{11}})\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}}$. Likewise, every element of $\mathbf{A}_{\mathbf{10}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{00}}^{-\mathbf{1}}$ is nonnegative. Therefore, the last term is nonnegative. The first term is nonnegative by Lemma 1. The content of the parentheses of the second term is:

$$
\mathbf{A}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{01}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{10}}^{-\mathbf{1}} + \mathbf{A}_{\mathbf{00}}^{-\mathbf{1}} = \mathbf{A}_{\mathbf{11}}^{-\mathbf{1}}(\mathbf{A}_{\mathbf{01}} - \mathbf{A}_{\mathbf{11}})\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{10}}^{-\mathbf{1}}(\mathbf{A}_{\mathbf{00}} - \mathbf{A}_{\mathbf{10}})\mathbf{A}_{\mathbf{00}}^{-\mathbf{1}}.
$$

As

$$
\mathbf{A}_{\mathbf{01}} - \mathbf{A}_{\mathbf{11}} = \left(\frac{\theta-1}{\theta}\right)^{\theta}(\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{10}})\mathbf{W}' = \left(\frac{\theta-1}{\theta}\right)^{\theta}(\mathbf{E}_{\mathbf{10}} - \mathbf{E}_{\mathbf{00}})\mathbf{W}' = \mathbf{A}_{\mathbf{00}} - \mathbf{A}_{\mathbf{10}} \equiv \mathbf{C},
$$

the previous expression is equal to:

$$
\begin{aligned}
&(\mathbf{A}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{10}}^{-\mathbf{1}})\mathbf{C}\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}} + \mathbf{A}_{\mathbf{01}}^{-\mathbf{1}}\mathbf{C}(\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}} - \mathbf{A}_{\mathbf{00}}^{-\mathbf{1}}) \\
&= \mathbf{A}_{\mathbf{11}}^{-\mathbf{1}}(\mathbf{A}_{\mathbf{10}} - \mathbf{A}_{\mathbf{11}})\mathbf{A}_{\mathbf{10}}^{-\mathbf{1}}\mathbf{C}\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}} + \mathbf{A}_{\mathbf{01}}^{-\mathbf{1}}\mathbf{C}\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}}(\mathbf{A}_{\mathbf{00}} - \mathbf{A}_{\mathbf{01}})\mathbf{A}_{\mathbf{00}}^{-\mathbf{1}}.
\end{aligned}
$$

Each element of this expression is nonnegative by (32). Thus, the third term is nonnegative. These observations establish that term (35) is nonnegative.

Term (36) is expanded as follows.

$$
\begin{aligned}
&\frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{11}} - \mathbf{P}_{\theta\mathbf{01}})\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}}\boldsymbol{\Phi}\mathbf{e}_{\mathbf{01}} - \frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{10}} - \mathbf{P}_{\theta\mathbf{00}})\mathbf{A}_{\mathbf{00}}^{-\mathbf{1}}\boldsymbol{\Phi}\mathbf{e}_{\mathbf{00}} \\
&= \frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{11}} - \mathbf{P}_{\theta\mathbf{01}} - \mathbf{P}_{\theta\mathbf{10}} + \mathbf{P}_{\theta\mathbf{00}})\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}}\boldsymbol{\Phi}\mathbf{e}_{\mathbf{01}} + \frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{10}} - \mathbf{P}_{\theta\mathbf{00}})\mathbf{A}_{\mathbf{01}}^{-\mathbf{1}}(\mathbf{A}_{\mathbf{00}} - \mathbf{A}_{\mathbf{01}})\mathbf{A}_{\mathbf{00}}^{-\mathbf{1}}\boldsymbol{\Phi}\mathbf{e}_{\mathbf{01}} \\
&+ \frac{1}{\theta}(\mathbf{P}_{\theta\mathbf{10}} - \mathbf{P}_{\theta\mathbf{00}})\mathbf{A}_{\mathbf{00}}^{-\mathbf{1}}\boldsymbol{\Phi}(\mathbf{e}_{\mathbf{01}} - \mathbf{e}_{\mathbf{00}}).
\end{aligned}
$$

The second term is nonnegative by Lemmas 1 and (32). The third term is nonnegative by Lemma

1. The content of the parentheses of the first term after substituting definition (14) is:

$$
\begin{aligned}
&(\mathbf{P}_{\theta\mathbf{11}} - \mathbf{P}_{\theta\mathbf{01}} - \mathbf{P}_{\theta\mathbf{10}} + \mathbf{P}_{\theta\mathbf{00}})\mathbf{1} \\
&\propto \Bigg[\mathbf{B}_{\mathbf{11}}^{-\mathbf{1}}(\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}}) - \mathbf{B}_{\mathbf{10}}^{-\mathbf{1}}(\mathbf{E}_{\mathbf{10}} - \mathbf{E}_{\mathbf{00}}) + \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{B}_{\mathbf{11}}^{-\mathbf{1}}(\mathbf{E}_{\mathbf{11}}\mathbf{W}\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}}\mathbf{W}\mathbf{E}_{\mathbf{01}})\mathbf{B}_{\mathbf{01}}^{-\mathbf{1}} \\
&\quad + \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{B}_{\mathbf{10}}^{-\mathbf{1}}(\mathbf{E}_{\mathbf{10}}\mathbf{W}\mathbf{E}_{\mathbf{10}} - \mathbf{E}_{\mathbf{00}}\mathbf{W}\mathbf{E}_{\mathbf{00}})\mathbf{B}_{\mathbf{00}}^{-\mathbf{1}}\Bigg] \mathbf{w}_{\mathbf{0}} \\
&= \big[(\mathbf{B}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{B}_{\mathbf{10}}^{-\mathbf{1}})(\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}}) \\
&\quad + \left(\frac{\theta-1}{\theta}\right)^{\theta-1} (\mathbf{B}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{B}_{\mathbf{10}}^{-\mathbf{1}})(\mathbf{E}_{\mathbf{11}}\mathbf{W}\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}}\mathbf{W}\mathbf{E}_{\mathbf{01}})\mathbf{B}_{\mathbf{01}}^{-\mathbf{1}} \\
&\quad + \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{B}_{\mathbf{10}}^{-\mathbf{1}}(\mathbf{E}_{\mathbf{11}}\mathbf{W}\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}}\mathbf{W}\mathbf{E}_{\mathbf{01}} - \mathbf{E}_{\mathbf{10}}\mathbf{W}\mathbf{E}_{\mathbf{10}} + \mathbf{E}_{\mathbf{00}}\mathbf{W}\mathbf{E}_{\mathbf{00}})\mathbf{B}_{\mathbf{01}}^{-\mathbf{1}} \\
&\quad + \left(\frac{\theta-1}{\theta}\right)^{\theta-1} \mathbf{B}_{\mathbf{10}}^{-\mathbf{1}}(\mathbf{E}_{\mathbf{10}}\mathbf{W}\mathbf{E}_{\mathbf{10}} - \mathbf{E}_{\mathbf{00}}\mathbf{W}\mathbf{E}_{\mathbf{00}})(\mathbf{B}_{\mathbf{01}}^{-\mathbf{1}} - \mathbf{B}_{\mathbf{00}}^{-\mathbf{1}})\Bigg] \mathbf{w}_{\mathbf{0}}.
\end{aligned}
$$

We use the fact that $\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}} = \mathbf{E}_{\mathbf{10}} - \mathbf{E}_{\mathbf{00}}$ in the last equality. Every element of $\mathbf{B}_{\mathbf{11}}^{-\mathbf{1}} - \mathbf{B}_{\mathbf{10}}^{-\mathbf{1}} = \mathbf{B}_{\mathbf{11}}^{-\mathbf{1}}(\mathbf{B}_{\mathbf{10}} - \mathbf{B}_{\mathbf{11}})\mathbf{B}_{\mathbf{10}}^{-\mathbf{1}}$ in the first and second terms in the last expression is nonnegative by (31). Likewise, every element of $\mathbf{B}_{\mathbf{01}}^{-\mathbf{1}} - \mathbf{B}_{\mathbf{00}}^{-\mathbf{1}}$ in the last term is nonnegative.

Every element of $\mathbf{E}_{\mathbf{11}}\mathbf{W}\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}}\mathbf{W}\mathbf{E}_{\mathbf{01}}$ in the second term and $\mathbf{E}_{\mathbf{10}}\mathbf{W}\mathbf{E}_{\mathbf{10}} - \mathbf{E}_{\mathbf{00}}\mathbf{W}\mathbf{E}_{\mathbf{00}}$ in the last term is nonnegative by (31). As $\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}} = \mathbf{E}_{\mathbf{10}} - \mathbf{E}_{\mathbf{00}}$, the contents of the parentheses in the third term can be rearranged as follows:

$$
\begin{aligned}
&\mathbf{E}_{\mathbf{11}}\mathbf{W}\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}}\mathbf{W}\mathbf{E}_{\mathbf{01}} - \mathbf{E}_{\mathbf{10}}\mathbf{W}\mathbf{E}_{\mathbf{10}} + \mathbf{E}_{\mathbf{00}}\mathbf{W}\mathbf{E}_{\mathbf{00}} \\
&= (\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}})\mathbf{W}(\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{10}}) + (\mathbf{E}_{\mathbf{01}} - \mathbf{E}_{\mathbf{00}})\mathbf{W}(\mathbf{E}_{\mathbf{11}} - \mathbf{E}_{\mathbf{01}}).
\end{aligned}
$$

Every element of the last expression is nonnegative. Thus, the first term of the expanded (36) is nonnegative. These observations establish that expression (36) is nonnegative.

Thus, we have established that (33) is nonnegative.

Premultiplying (33) by $\mathbf{e}_{\mathbf{b}}'$ gives $\Pi_b(\mathbf{e}_{\mathbf{11}}) - \Pi_b(\mathbf{e}_{\mathbf{01}}) - \{\Pi_b(\mathbf{e}_{\mathbf{10}}) - \Pi_b(\mathbf{e}_{\mathbf{00}})\}$. This is nonnegative if (33) is nonnegative. □

Appendix 2: Proofs of Propositions 1 and 2

Proof of Proposition 1. The domain and region of map (20) are defined on the complete lattice by Lemma 4. The map is monotonically increasing (order-preserving) by Lemma 5. Therefore, there exists a fixed point $\mathbf{e}^* = \mathcal{G}(\mathbf{e}^*)$ by Tarski's fixed point theorem (Theorem 1 in Tarski, 1955). □

Proof of Proposition 2. We apply the following theorem:

Theorem 2 in Zhou (1994): The set of Nash equilibria of a supermodular game G is a complete *lattice.*

All we must show is that the game is supermodular, i.e., i) each bank's profit is supermodular with respect to the bank's strategy $\mathbf{e_b}$, and ii) each bank's profit has increasing differences in $\mathbf{e_b}$.
i) First, we note $\mathbf{e_b} \vee \mathbf{g_b} - \mathbf{e_b} \wedge \mathbf{g_b} = \mathbf{e_b} + \mathbf{g_b} - \mathbf{e_b} \wedge \mathbf{g_b} - \mathbf{e_b} \wedge \mathbf{g_b}$, i.e.,

$$\mathbf{e_b} \vee \mathbf{g_b} - \mathbf{e_b} = \mathbf{g_b} - \mathbf{e_b} \wedge \mathbf{g_b}, \tag{37}$$

where $\mathbf{g_b}$ is an $n \times 1$ vector whose i-th element g_i is 0 or 1, and $g_i = 0$ if $i \notin D_b^c$. We also note that

$$\begin{aligned} \mathbf{e_b} \vee \mathbf{g_b} \geq \mathbf{e_b} \geq \mathbf{e_b} \wedge \mathbf{g_b}, \\ \mathbf{e_b} \vee \mathbf{g_b} \geq \mathbf{g_b} \geq \mathbf{e_b} \wedge \mathbf{g_b}. \end{aligned} \tag{38}$$

From Lemma 5 and (38), it is readily shown that

$$\Pi_b(\mathbf{e_b} \vee \mathbf{g_b}; \mathbf{e_{b'}}) - \Pi_b(\mathbf{e_b}; \mathbf{e_{b'}}) \geq \Pi_b(\mathbf{g_b}; \mathbf{e_{b'}}) - \Pi_b(\mathbf{e_b} \wedge \mathbf{g_b}; \mathbf{e_{b'}}).$$

We can rewrite this by (37) as follows:

$$\begin{aligned} &\left\{\Pi_b(\mathbf{e_b} \vee \mathbf{g_b}; \mathbf{e_{b'}}) - \mathbf{f}'(\mathbf{e_b} \vee \mathbf{g_b})\right\} - \left\{\Pi_b(\mathbf{e_b}; \mathbf{e_{b'}}) - \mathbf{f}'\mathbf{e_b}\right\} \\ &\geq \left\{\Pi_b(\mathbf{g_b}; \mathbf{e_{b'}}) - \mathbf{f}'\mathbf{g_b}\right\} - \left\{\Pi_b(\mathbf{e_b} \wedge \mathbf{g_b}; \mathbf{e_{b'}}) - \mathbf{f}'(\mathbf{e_b} \wedge \mathbf{g_b})\right\}, \end{aligned}$$

where $\mathbf{f}$ is the vector of newly required fund f_i for firm i to continue its business. Thus, we have established the supermodularity of each bank's profit with respect to its strategy space $\mathbf{e_b}$.
ii) From Lemma 5, if $\mathbf{e_b} \geq \mathbf{g_b}$ and $\mathbf{v_{b'}} \geq \mathbf{w_{b'}}$,

$$\Pi_b(\mathbf{e_b}; \mathbf{v_{b'}}) - \Pi_b(\mathbf{g_b}; \mathbf{v_{b'}}) \geq \Pi_b(\mathbf{e_b}; \mathbf{w_{b'}}) - \Pi_b(\mathbf{g_b}; \mathbf{w_{b'}}).$$

By rearranging this expression, we obtain

$$\begin{aligned} &\left\{\Pi_b(\mathbf{e_b}; \mathbf{v_{b'}}) - \mathbf{f}'\mathbf{e_b}\right\} - \left\{\Pi_b(\mathbf{e_b}; \mathbf{w_{b'}}) - \mathbf{f}'\mathbf{e_b}\right\} \\ &\geq \left\{\Pi_b(\mathbf{g_b}; \mathbf{v_{b'}}) - \mathbf{f}'\mathbf{g_b}\right\} - \left\{\Pi_b(\mathbf{g_b}; \mathbf{w_{b'}}) - \mathbf{f}'\mathbf{g_b}\right\}. \end{aligned}$$

Thus, the bank profit function exhibits increasing differences. □

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